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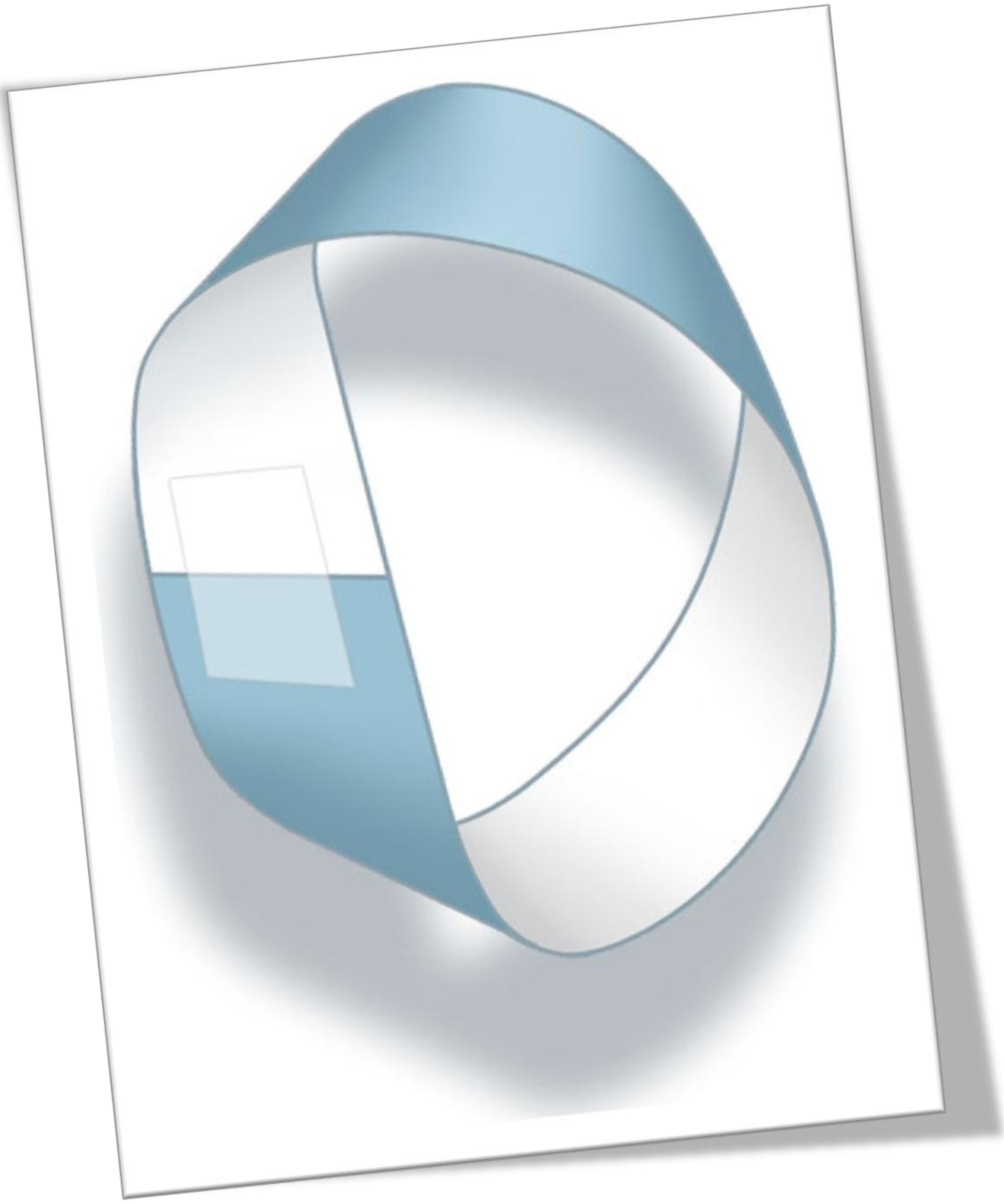
THE MÖBIUS STRIP



Author:
Coilin P. Boylan Jeritslev – s082758

Advisers:
Steen Markvorsen & Jens Gravesen

Contests

Introduction

In the project we are searching after an exact definition of a Möbius Strip and its unique properties. In a simple case we can create the Möbius Strip, if we take a strip of paper, turn it 180 degrees, and glue the ends together. But how do we define the shape mathematically? Which properties we need before we can classify a given shape as a Möbius Strip? Has the strip a specific value of Gaussian curvature?

We try also to prove mathematically that the Möbius Strip can fold out like a two-dimensional plane in a space. We know that the shape is determined by the bending energy is minimal, and although no one knows the exact shape, one can find an approximation by using the differential geometry and optimization. We want to investigate about the bending energy of the strip is really minimal and so illustrated the bending energy in the strip like the figure below. From the figure 1 we can assume that the bending energy becomes larger, if the width of the strip is larger.

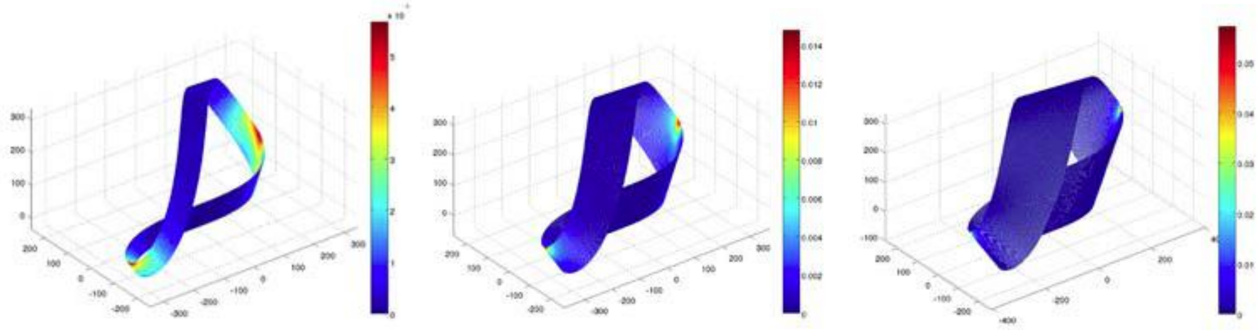


Figure 1 The bending energy in the Möbius strip with three different widths.

From there we are interesting to know, how wide the strip may be without to intersect. Have the length of the mean curve and the number of rotation round its mean curve an impact on how wide the strip may be?

For these tasks we use Maple and MATLAB to illustrate and solve the problems.

Have a good reading!

Theory

In the following we consider the Euclidean 3-space.

The Möbius Strip as a Ruled Surface

A plane piece of paper can be wrapped on a cylinder in the obvious way without crumpling the paper. If we draw a curve on the plane, then after wrapping it becomes a curve on the cylinder. Because there is no crumpling, the lengths of these two curves will be the same.

It means that we will observed that a plane and a generalized cylinder, when suitably parameterized, have the same first fundamental form, since the lengths are computed as the integral of the first fundamental form. The first fundamental forms of the two surfaces are the same.

Let $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}\big(u(t), v(t)\big)$ be a curve in the surface patch $\boldsymbol{\sigma}$, its arc-length starting at a point $\boldsymbol{\gamma}(t_0)$ is given by

$$s = \int_{t_0}^{t} \|\dot{\boldsymbol{\gamma}}(\tau)\| d\tau$$

By the chain rule, $\dot{\boldsymbol{\gamma}} = \boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}$, we get

$$\begin{aligned}
\|\dot{\boldsymbol{\gamma}}\|^2 &= \langle \dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}} \rangle \\
&= \langle \boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}, \boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v} \rangle \\
&= \langle \boldsymbol{\sigma}_u \dot{u}, \boldsymbol{\sigma}_u \dot{u} \rangle + \langle \boldsymbol{\sigma}_u \dot{u}, \boldsymbol{\sigma}_v \dot{v} \rangle + \langle \boldsymbol{\sigma}_v \dot{v}, \boldsymbol{\sigma}_u \dot{u} \rangle + \langle \boldsymbol{\sigma}_v \dot{v}, \boldsymbol{\sigma}_v \dot{v} \rangle \\
&= \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_u \rangle \dot{u}^2 + \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle \dot{u}\dot{v} + \langle \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_u \rangle \dot{v}\dot{u} + \langle \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_v \rangle \dot{v}^2 \\
&= \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_u \rangle \dot{u}^2 + 2\langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle \dot{u}\dot{v} + \langle \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_v \rangle \dot{v}^2 \\
&= E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2
\end{aligned}$$

Where

$$E = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_u \rangle = \|\boldsymbol{\sigma}_u\|^2, \qquad F = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle, \qquad G = \langle \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_v \rangle = \|\boldsymbol{\sigma}_v\|^2$$

Therefore the length of the curve is

$$s = \int_{t_0}^{t} \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} d\tau$$

If we substitute the $d\tau$ inside the square root with

$$\begin{aligned}
\left(\frac{du}{d\tau}\right)^2 \cdot (d\tau)^2 &= du^2 \\
\frac{du}{d\tau} \cdot \frac{dv}{d\tau} \cdot (d\tau)^2 &= du dv \\
\left(\frac{dv}{d\tau}\right)^2 \cdot (d\tau)^2 &= dv^2
\end{aligned}$$

So we write

$$s = \int_{t_0}^{t} \sqrt{E\frac{du^2}{(d\tau)^2} + 2F\frac{dudv}{(d\tau)^2} + G\frac{dv^2}{(d\tau)^2}}d\tau$$
$$= \int_{t_0}^{t} \sqrt{Edu^2 + 2Fdudv + Gdv^2}$$

We see that s is the integral of the square root of the expression, called the first fundamental form of $\boldsymbol{\sigma}$,

$$Edu^2 + 2Fdudv + Gdv^2.$$

The first fundamental form will change when the surface patch is changed.

Example 1

A generalized plane in $\mathbb{R}^3$ can define by

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{a} + u\boldsymbol{p} + v\boldsymbol{q}$$

With $\boldsymbol{p}$ and $\boldsymbol{q}$ being perpendicular unit vectors, we have $\boldsymbol{\sigma}_u = \boldsymbol{p}$ and $\boldsymbol{\sigma}_v = \boldsymbol{q}$, so

$$E = \|\boldsymbol{\sigma}_u\|^2 = \|\boldsymbol{p}\|^2 = 1, \qquad F = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle = \langle \boldsymbol{p}, \boldsymbol{q} \rangle = 0, \qquad G = \|\boldsymbol{\sigma}_v\|^2 = \|\boldsymbol{q}\|^2 = 1.$$

Hence the first fundamental form of a plane is simply

$$Edu^2 + 2Fdudv + Gdv^2 = du^2 + dv^2$$

Example 2

A generalized cylinder in $\mathbb{R}^3$ can define by

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{p}$$

With $\boldsymbol{\gamma}$ being an unit-speed and $\boldsymbol{p}$ is a unit vector. We can assume that $\boldsymbol{\gamma}$ is contained in a plane perpendicular to $\boldsymbol{p}$. Then we have $\boldsymbol{\sigma}_u = \boldsymbol{\gamma}_u$ and $\boldsymbol{\sigma}_v = \boldsymbol{p}$, so

$$E = \|\boldsymbol{\sigma}_u\|^2 = \|\boldsymbol{\gamma}_u\|^2 = 1, \qquad F = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle = \langle \boldsymbol{\gamma}_u, \boldsymbol{p} \rangle = 0, \qquad G = \|\boldsymbol{\sigma}_v\|^2 = \|\boldsymbol{p}\|^2 = 1.$$

Hence the first fundamental form of a cylinder is simply

$$Edu^2 + 2Fdudv + Gdv^2 = du^2 + dv^2$$

Note that we have proven that first fundamental forms of the two surfaces, a generalized plane and a generalized cylinder, are the same. We have the definition

Definition 1

If $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are surfaces, a diffeomorphism $f: \boldsymbol{\sigma}_1 \to \boldsymbol{\sigma}_2$ is called an isometry if every curve in $\boldsymbol{\sigma}_1$ transforming to curves in $\boldsymbol{\sigma}_2$ have the same length, i.e. the same first fundamental form. We say that $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are isometric, if the mapping f is isometry.

Now we use the definition and the generalized ruled surface to find when a Möbius strip is isometric to a generalized plane.

From [1] (page 1) we know that a generalized ruled surface in $\mathbb{R}^3$ is defined by

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\alpha}(u) + v\boldsymbol{w}(u)$$

With the base curve $\boldsymbol{\alpha}: I \to \mathbb{R}^3$ and the director curve $\boldsymbol{w}: I \to \mathbb{R}^3 \backslash \{\boldsymbol{0}\}$ and the scalar v that generates the rulings. The Möbius strip belongs to the characterization, since the strip is made by a flat paper and the paper can be generated by the ruled surface.

Note if $\boldsymbol{w}'(u) = \boldsymbol{0}$ for all $u \in I$ we can reduce the expression such that $\boldsymbol{w}$ is a constant speed vector, i.e.:

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\alpha}(u) + v\boldsymbol{w}$$

The expression is like to the characterization of a cylinder in $\mathbb{R}^3$. Therefore the phenomenal is called a cylindrical ruled surface.

Let us assume a ruled surface. By simplifying the case we assume that $\boldsymbol{\alpha}(u)$ and $\boldsymbol{w}(u)$ are respectively an unit speed vector and an unit vector, i.e. $\|\boldsymbol{\alpha}'(u)\| = 1$ and $\|\boldsymbol{w}(u)\| = 1$, because this assumptions do not give a loss of generality. Notice that by the chain rule the assumption $\|\boldsymbol{w}(u)\| = 1$ implies that $\langle \boldsymbol{w}'(u), \boldsymbol{w}(u)\rangle = 0$ for all $u \in I$.

$$\begin{aligned}
\frac{d}{du}(\|\boldsymbol{w}(u)\|) &= \frac{d}{du}(\langle \boldsymbol{w}(u), \boldsymbol{w}(u)\rangle) \\
&= \langle \frac{d\boldsymbol{w}(u)}{du}, \boldsymbol{w}(u)\rangle + \langle \boldsymbol{w}(u), \frac{d\boldsymbol{w}(u)}{du}\rangle \\
&= 2\,\langle \frac{d\boldsymbol{w}(u)}{du}, \boldsymbol{w}(u)\rangle = 0
\end{aligned}$$

We find the first fundamental form of the generalized surface:

$$\begin{aligned}
E &= \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_u \rangle \\
&= \langle \boldsymbol{\alpha}' + v\cdot\boldsymbol{w}', \boldsymbol{\alpha}' + v\cdot\boldsymbol{w}'\rangle \\
&= \langle \boldsymbol{\alpha}', \boldsymbol{\alpha}' + v\cdot\boldsymbol{w}'\rangle + v\cdot\langle \boldsymbol{w}', \boldsymbol{\alpha}' + v\cdot\boldsymbol{w}'\rangle \\
&= \langle \boldsymbol{\alpha}', \boldsymbol{\alpha}'\rangle + 2v\cdot\langle \boldsymbol{\alpha}', \boldsymbol{w}'\rangle + v^2\cdot\langle \boldsymbol{w}', \boldsymbol{w}'\rangle \\
&= \|\boldsymbol{\alpha}'\|^2 + 2v\cdot\langle \boldsymbol{\alpha}', \boldsymbol{w}'\rangle + v^2\cdot\|\boldsymbol{w}'\|^2 \\
F &= \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle \\
&= \langle \boldsymbol{\alpha}' + v\cdot\boldsymbol{w}', \boldsymbol{w}\rangle \\
&= \langle \boldsymbol{\alpha}', \boldsymbol{w}\rangle + v\cdot\langle \boldsymbol{w}', \boldsymbol{w}\rangle \\
G &= \langle \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_v \rangle \\
&= \langle \boldsymbol{w}, \boldsymbol{w}\rangle \\
&= \|\boldsymbol{w}\|^2
\end{aligned}$$

The ruled surface with the unit vector $\boldsymbol{w}$ and the unit speed vector $\boldsymbol{\alpha}$ gives

$$E = 1 + 2v\cdot\langle \boldsymbol{\alpha}', \boldsymbol{w}'\rangle + v^2\cdot\|\boldsymbol{w}'\|^2, \qquad F = \langle \boldsymbol{\alpha}', \boldsymbol{w}\rangle, \qquad G = 1.$$

The first fundamental form of a generalized plane in $\mathbb{R}^3$ is

$$Edu^2 + 2Fdudv + Gdv^2 = du^2 + dv^2$$

<u>Notice!</u> It is important to choose the parameterization carefully; the vectors that span the plan in $\mathbb{R}^3$have to be unit orthonormal vectors. If the vectors are not unit or orthonormal – like the two last cases in the illustration below – then we do **<u>not</u>** get the first fundamental form $E = 1,\ F = 0,\ G = 1$. See the two examples to understand why.

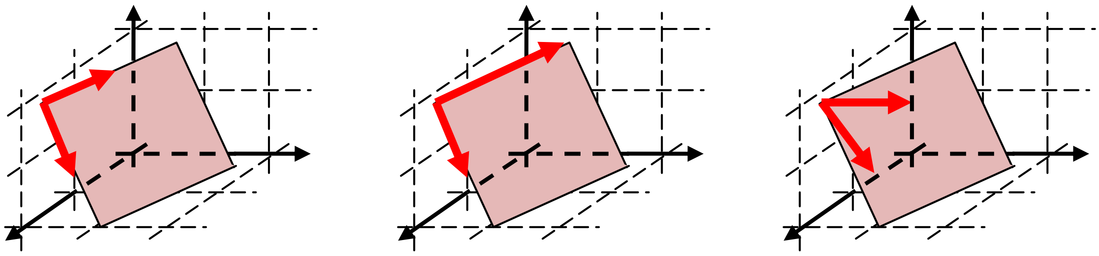


Figure 2 Span the space $\mathbb{R}^3$ with unit orthonormal basis (left), non-unit orthonormal basis (center) and unit non-orthonormal basis (right).

Example 3

Let a plane in $\mathbb{R}^3$ be define by

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{a} + u\boldsymbol{p} + v\boldsymbol{q}$$

With $\boldsymbol{p}$ and $\boldsymbol{q}$ being perpendicular **<u>non</u>**-unit vectors, we have $\boldsymbol{\sigma}_u = \boldsymbol{p}$ and $\boldsymbol{\sigma}_v = \boldsymbol{q}$, so

$$E = \|\boldsymbol{\sigma}_u\|^2 = \|\boldsymbol{p}\|^2 \neq 1, \qquad F = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle = \langle \boldsymbol{p}, \boldsymbol{q} \rangle = 0, \qquad G = \|\boldsymbol{\sigma}_v\|^2 = \|\boldsymbol{q}\|^2 \neq 1.$$

Hence the first fundamental form of a plane is

$$Edu^2 + 2Fdudv + Gdv^2 \neq du^2 + dv^2$$

Example 4

Let a plane in $\mathbb{R}^3$ be define by

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{a} + u\boldsymbol{p} + v\boldsymbol{q}$$

With $\boldsymbol{p}$ and $\boldsymbol{q}$ being **<u>non</u>**-perpendicular unit vectors, we have $\boldsymbol{\sigma}_u = \boldsymbol{p}$ and $\boldsymbol{\sigma}_v = \boldsymbol{q}$, so

$$E = \|\boldsymbol{\sigma}_u\|^2 = \|\boldsymbol{p}\|^2 = 1, \qquad F = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle = \langle \boldsymbol{p}, \boldsymbol{q} \rangle \neq 0, \qquad G = \|\boldsymbol{\sigma}_v\|^2 = \|\boldsymbol{q}\|^2 = 1.$$

Hence the first fundamental form of a plane is

$$Edu^2 + 2Fdudv + Gdv^2 \neq du^2 + dv^2$$

Therefore unless otherwise stated we shall assume that every Möbius strip in the space is suitably parameterized. There exists an example where a Möbius strip is not correctly parameterized. We can illustrate the case.

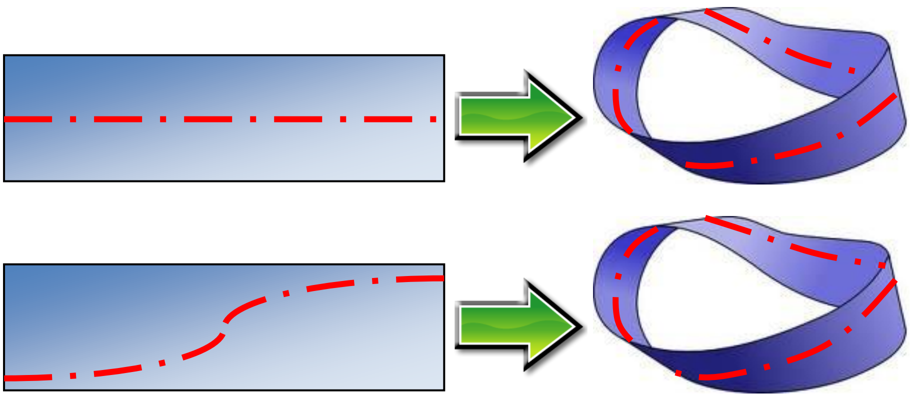


Figure 3 The upper case is the correctly parameterized Möbius strip, while the lower case is the incorrectly parameterized Möbius strip. The red line is the base curve.

A Möbius strip which is correctly parameterized and is isometric to the generalized plan is called a flat Möbius strip. But if the strip doesn't satisfy the condition although it is correctly parameterized, the strip is not a flat Möbius strip. So the flat Möbius strip belonging to the generalized ruled surface, is according to definition 1 isometric to the plane if and only if it satisfies

$$E = 1 + 2v \cdot \langle \boldsymbol{\alpha}', \boldsymbol{w}' \rangle + v^2 \cdot \|\boldsymbol{w}'\|^2 = 1, \qquad F = \langle \boldsymbol{\alpha}', \boldsymbol{w} \rangle = 0, \qquad G = 1.$$

For all u and v.

The geometrical interpretation of F is the Möbius strip is flat if $\boldsymbol{\alpha}'$ and $\boldsymbol{w}'$ both are perpendicular to $\boldsymbol{w}$, because we have $\langle \boldsymbol{\alpha}', \boldsymbol{w} \rangle = 0$ in F and $\langle \boldsymbol{w}', \boldsymbol{w} \rangle = 0$ as a consequence of $\boldsymbol{w}$ is unit vector, $\boldsymbol{\alpha}'$ is in the same plane as $\boldsymbol{w}'$. See the illustration below.

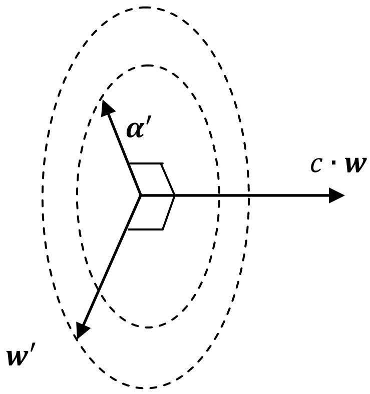


Figure 4 A simple system of vectors in the ruled surface for any Möbius strips.

The equation of E leads us to

$$0 = 2v \cdot \langle \boldsymbol{\alpha}', \boldsymbol{w}' \rangle + v^2 \cdot \|\boldsymbol{w}'\|^2$$
$$\Rightarrow 2\langle \boldsymbol{\alpha}', \boldsymbol{w}' \rangle + v \cdot \langle \boldsymbol{w}', \boldsymbol{w}' \rangle = 0$$

Hence

$$\langle 2\boldsymbol{\alpha}', \boldsymbol{w}' \rangle = \langle -v \cdot \boldsymbol{w}', \boldsymbol{w}' \rangle$$

Therefore if we want to use the ruled surface to describe the flat Möbius strip we need to satisfy the following conditions

$$\langle 2\boldsymbol{\alpha}', \boldsymbol{w}' \rangle = \langle -v \cdot \boldsymbol{w}', \boldsymbol{w}' \rangle \text{ and } \langle \boldsymbol{\alpha}', \boldsymbol{w} \rangle = 0$$

With $\|\boldsymbol{\alpha}'(u)\| = 1$ and $\|\boldsymbol{w}(u)\| = 1$.

Example 5

Let a simply band in $\mathbb{R}^3$ be defined by

$$\boldsymbol{\sigma}(u,v) = \begin{pmatrix} \cos(u) \\ \sin(u) \\ 0 \end{pmatrix} + v \cdot \begin{pmatrix} \cos\left(\frac{u}{2}\right) \cdot \cos(u) \\ \cos\left(\frac{u}{2}\right) \cdot \sin(u) \\ \sin\left(\frac{u}{2}\right) \end{pmatrix}$$

With $u \in [0,2\pi]$ and $v \in [-L, L]$ for $L \in \mathbb{R}$. The figure of the band is illustrated in the figure 2.

We obtain

$$\boldsymbol{\sigma}_u = \begin{pmatrix} -\sin(u) \\ \cos(u) \\ 0 \end{pmatrix} + v \cdot \begin{pmatrix} -\frac{1}{2}\sin\left(\frac{u}{2}\right) \cdot \cos(u) - \cos\left(\frac{u}{2}\right) \cdot \sin(u) \\ -\frac{1}{2}\sin\left(\frac{u}{2}\right) \cdot \sin(u) + \cos\left(\frac{u}{2}\right) \cdot \cos(u) \\ \frac{1}{2}\cos\left(\frac{u}{2}\right) \end{pmatrix} \text{ and } \boldsymbol{\sigma}_v = \begin{pmatrix} \cos\left(\frac{u}{2}\right) \cdot \cos(u) \\ \cos\left(\frac{u}{2}\right) \cdot \sin(u) \\ \sin\left(\frac{u}{2}\right) \end{pmatrix},$$

so

$$E = \|\boldsymbol{\sigma}_u\|^2 = 1 + 2v \cdot \cos\left(\frac{u}{2}\right) + v^2 \cdot \left(\frac{1}{4} + \cos^2\left(\frac{u}{2}\right)\right), \qquad F = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle = 0, \qquad G = \|\boldsymbol{\sigma}_v\|^2 = 1.$$

The first fundamental form of a generalized plane is

$$Edu^2 + 2Fdudv + Gdv^2 = du^2 + dv^2$$

Since $2v \cdot \cos\left(\frac{u}{2}\right) + v^2 \cdot \left(\frac{1}{4} + \cos^2\left(\frac{u}{2}\right)\right)$ does not give zero for all $u \in [0,2\pi]$ and $v \in [-L, L]$ for $L \in \mathbb{R}$, the first fundamental form of the band is not the same as the generalized plane. Therefore the band is **not** a **flat** Möbius strip.

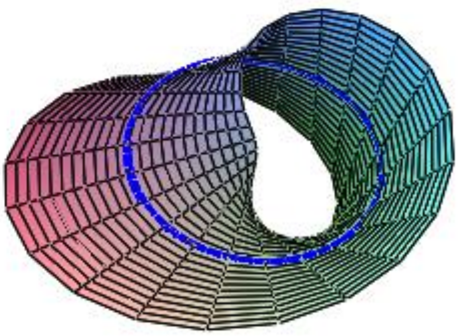


Figure 5 The “non-flat” Möbius band from Example 5, where the blue line in the band is the base curve.

This example shows that it is not always easy to judge a strip based on the view of the geometrical figure, about a Möbius strip is flat or not. Every view can mislead. We can just claim that any not-orientable ruled surface, which has a closed base curve, is a Möbius strip, but we cannot be sure that the strip is flat.

It can be a little hard to find the correct examples based on the information $\langle 2\boldsymbol{\alpha}', \boldsymbol{w}' \rangle = \langle -v \cdot \boldsymbol{w}', \boldsymbol{w}' \rangle$ and $\langle \boldsymbol{\alpha}', \boldsymbol{w} \rangle = 0$, then we still need to improve the hypothesis. To be able to improve the hypothesis, we need a new definition.

Definition 2

Let $\boldsymbol{\sigma}(u, v)$ be a surface patch with first and second fundamental forms

$$Edu^2 + 2Fdudv + Gdv^2 \text{ and } Ldu^2 + 2Mdudv + Ndv^2$$

Respectively, where $L = \langle \boldsymbol{N}_{\boldsymbol{\sigma}}, \boldsymbol{\sigma}_{uu} \rangle, N = \langle \boldsymbol{N}_{\boldsymbol{\sigma}}, \boldsymbol{\sigma}_{vv} \rangle$ and $M = \langle \boldsymbol{N}_{\boldsymbol{\sigma}}, \boldsymbol{\sigma}_{uv} \rangle$ with

$$\boldsymbol{N}_{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|}$$

Then the Gaussian curvature is

$$K = \frac{LN - M^2}{EG - F^2}$$

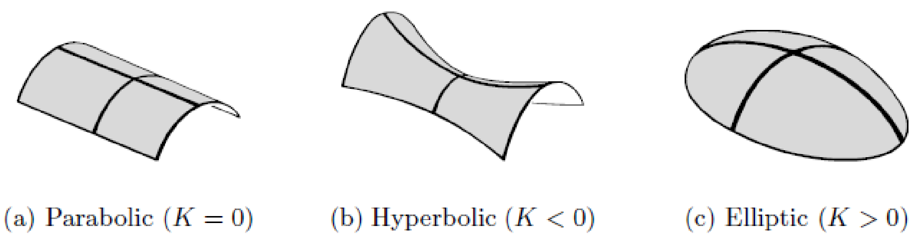


Figure 6 The geometrical interpret of the Gaussian curvature.

From [1] (page 8) the Gaussian curvature of a ruled surface satisfies $K \leq 0$ everywhere.The figures above show that any ruled surface without cluster points can only be either a cylinder or a hyperboloid. Here we can see the usefulness of the Gaussian curvature.

The Möbius strip has the same first fundamental form as the generalized plane and is a cylindrical ruled surface. By this way we conclude that the Gaussian curvature of the strip has to satisfy $K = 0$, which means that we have to find a ruled surface that according to definition 2 satisfies $LN - M^2 = 0$.

Let the ruled surface be defined by $\boldsymbol{\sigma}(u, v) = \boldsymbol{\alpha}(u) + v \cdot \boldsymbol{w}(u)$, so

$$\begin{aligned}
\boldsymbol{\sigma}_u &= \boldsymbol{\alpha}' + v \cdot \boldsymbol{w}', \\
\boldsymbol{\sigma}_v &= \boldsymbol{w}, \\
\boldsymbol{\sigma}_t \times \boldsymbol{\sigma}_v &= (\boldsymbol{\alpha}' + v \cdot \boldsymbol{w}') \times \boldsymbol{w} \\
&= \boldsymbol{\alpha}' \times \boldsymbol{w} + v \cdot \boldsymbol{w}' \times \boldsymbol{w}, \\
\boldsymbol{\sigma}_{uu} &= \boldsymbol{\alpha}'' + v \cdot \boldsymbol{w}'', \\
\boldsymbol{\sigma}_{uv} &= \boldsymbol{w}', \\
\boldsymbol{\sigma}_{vv} &= \boldsymbol{0}
\end{aligned}$$

The standard unit normal of the surface is

$$\boldsymbol{N}_\sigma = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} = \frac{\boldsymbol{\alpha}' \times \boldsymbol{w} + v \cdot \boldsymbol{w}' \times \boldsymbol{w}}{\|\boldsymbol{\alpha}' \times \boldsymbol{w} + v \cdot \boldsymbol{w}' \times \boldsymbol{w}\|}$$

The second fundamental form is:

$$\begin{aligned}
L &= \langle \boldsymbol{N}_\sigma, \boldsymbol{\sigma}_{uu} \rangle \\
&= \frac{\langle \boldsymbol{\alpha}' \times \boldsymbol{w} + v \cdot \boldsymbol{w}' \times \boldsymbol{w}, \boldsymbol{\alpha}'' + v \cdot \boldsymbol{w}'' \rangle}{\|\boldsymbol{\alpha}' \times \boldsymbol{w} + v \cdot \boldsymbol{w}' \times \boldsymbol{w}\|} \\
N &= \langle \boldsymbol{N}_\sigma, \boldsymbol{\sigma}_{vv} \rangle \\
&= 0 \\
M &= \langle \boldsymbol{N}_\sigma, \boldsymbol{\sigma}_{tv} \rangle \\
&= \frac{\langle \boldsymbol{\alpha}' \times \boldsymbol{w} + v \cdot \boldsymbol{w}' \times \boldsymbol{w}, \boldsymbol{w}' \rangle}{\|\boldsymbol{\alpha}' \times \boldsymbol{w} + v \cdot \boldsymbol{w}' \times \boldsymbol{w}\|} \\
&= \frac{\langle \boldsymbol{\alpha}' \times \boldsymbol{w}, \boldsymbol{w}' \rangle + v \cdot \langle \boldsymbol{w}' \times \boldsymbol{w}, \boldsymbol{w}' \rangle}{\|\boldsymbol{\alpha}' \times \boldsymbol{w} + v \cdot \boldsymbol{w}' \times \boldsymbol{w}\|} \\
&= \frac{\langle \boldsymbol{\alpha}' \times \boldsymbol{w}, \boldsymbol{w}' \rangle}{\|\boldsymbol{\alpha}' \times \boldsymbol{w} + v \cdot \boldsymbol{w}' \times \boldsymbol{w}\|}
\end{aligned}$$

Hence

$$LN - M^2 = -\left(\frac{\langle \boldsymbol{\alpha}' \times \boldsymbol{w}, \boldsymbol{w}' \rangle}{\|\boldsymbol{\alpha}' \times \boldsymbol{w} + v \cdot \boldsymbol{w}' \times \boldsymbol{w}\|} \right)^2 = 0$$

Which leads to the expression

$$\langle \boldsymbol{\alpha}' \times \boldsymbol{w}, \boldsymbol{w}' \rangle = 0$$

Note with the new expression we can improve the geometrical interpretation of the ruled surface.

As we know we have $\langle \boldsymbol{\alpha}', \boldsymbol{w} \rangle = 0$ and $\langle \boldsymbol{w}', \boldsymbol{w} \rangle = 0$ for the flat Möbius strip. Therefore we illustrate the figure below.

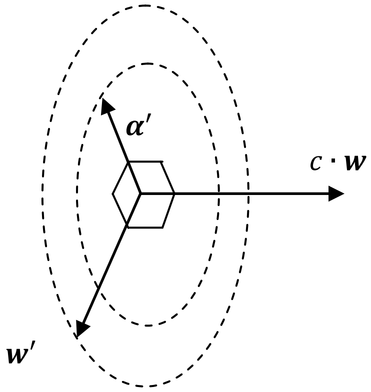


Figure 7 An improved system of vectors in the ruled surface for any flat Möbius strips.

The second way to find the Gaussian curvature equal to zero is using the distribution parameter λ and the line of striction $\boldsymbol{\beta}$.

Definition 3

Let the ruled surface in $\mathbb{R}^3$ be defined by $\boldsymbol{\sigma}(u, v) = \boldsymbol{\alpha}(u) + v \cdot \boldsymbol{w}(u)$ with the unit vector $\boldsymbol{w}$.

We call a curve $\boldsymbol{\beta}(u) \in \boldsymbol{\sigma}(u, v)$ the line of striction, if it satisfies $\langle \boldsymbol{\beta}'(u), \boldsymbol{w}'(u) \rangle = 0$.

With the parameter of the curve $\boldsymbol{\beta}(u) = \boldsymbol{\alpha}(u) + f(u) \cdot \boldsymbol{w}(u)$ for some real-value function $f = f(u)$, the line of striction of the surface is defined by:

$$\boldsymbol{\beta} = \boldsymbol{\alpha} - \frac{\langle \boldsymbol{\alpha}', \boldsymbol{w}' \rangle}{\langle \boldsymbol{w}', \boldsymbol{w}' \rangle} \cdot \boldsymbol{w}$$

If the surface is non-cylindrical, it has been known that there exists the line of strictions uniquely. Its points are called the central points of the ruled surface. If the surface is cylindrical, i.e. $\boldsymbol{w}' = \boldsymbol{0}$, we have the line expressed as $\boldsymbol{\beta} = \boldsymbol{\alpha}$.

Definition 4

Let the ruled surface in $\mathbb{R}^3$ be defined by $\boldsymbol{\sigma}(u, v) = \boldsymbol{\alpha}(u) + v \cdot \boldsymbol{w}(u)$ with the unit vector $\boldsymbol{w}$, so the distribution parameter for the ruled surface is expressed as

$$\lambda = \frac{\langle \boldsymbol{\beta}' \times \boldsymbol{w}, \boldsymbol{w}' \rangle}{\|\boldsymbol{w}'\|^2}$$

With the line of striction $\boldsymbol{\beta}$.

By using definition 3 and 4 we conclude that the distribution parameter of a cylindrical ruled surface is

$$\lambda = \frac{\langle \boldsymbol{\alpha}' \times \boldsymbol{w}, \boldsymbol{w}' \rangle}{\|\boldsymbol{w}'\|^2}$$

In any ruled surface the Gaussain curvature can according to [1] (page 8) write as

$$K = \frac{-\lambda^2}{(\lambda^2 + v^2)^2}$$

This show that the Gaussian curvature of a ruled surface satisfies $K \leq 0$. Since $K = 0$ if and only if $\lambda = 0$, i.e. K is zero only along those rulings which meet the line of striction. In the cylindrical ruled surface K is zero only along those rulings which meet the base curve. The distribution parameter is zero if and only if

$$\langle \boldsymbol{\alpha}' \times \boldsymbol{w}, \boldsymbol{w}' \rangle = 0$$

From [1] we have the theorem: Let $\boldsymbol{\sigma}(s, v) = \boldsymbol{\alpha}(s) + v \cdot \boldsymbol{w}(s)$ be a ruled surface with $\|\boldsymbol{w}(s)\| = 1$. Let $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + u(s) \cdot \boldsymbol{w}(s)$ be a curve on $\boldsymbol{\sigma}(s, v)$, where s is the arc-length of $\boldsymbol{\beta}(s)$. Consider the following three conditions on $\boldsymbol{\beta}(s)$:

- $\boldsymbol{\beta}(s)$ is a line of striction of $\boldsymbol{\sigma}(s, v)$.
- $\boldsymbol{\beta}(s)$ is a geodesic of $\boldsymbol{\sigma}(s, v)$.
- The angles between $\boldsymbol{\beta}'(s)$ and $\boldsymbol{w}(s)$ are constant.

If we assume that any two of above three conditions hold, then the other condition holds.

Since we have found that the base curve $\boldsymbol{\alpha}$ is a line of striction in the cylindrical ruled surface, and the angles between $\boldsymbol{\alpha}'$ and $\boldsymbol{w}$ are constant, we conclude that the base curve is a geodesic.

By using the proposition 8.2 in [2], where it is said that a curve on a surface is a geodesic, if and only if its geodesic curvature is zero everywhere, we conclude that the base curve in the cylindrical ruled surface has zero geodesic curvature everywhere.

Definition 5

Let $\boldsymbol{N}_{\boldsymbol{\sigma}}$ be the standard unit normal of the surface patch $\boldsymbol{\sigma}(u, v)$ given by

$$\boldsymbol{N}_{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|}$$

The normal curvature and the geodesic curvature of a unit-speed curve $\boldsymbol{\gamma}(s) = \boldsymbol{\sigma}\big(u(s), v(s)\big)$ in the surface patch, is defined by

$\kappa_n = \langle \ddot{\boldsymbol{\gamma}}, \boldsymbol{N}_{\boldsymbol{\sigma}} \rangle$ and $\kappa_g = \langle \ddot{\boldsymbol{\gamma}}, \boldsymbol{N}_{\boldsymbol{\sigma}} \times \dot{\boldsymbol{\gamma}} \rangle$

Respectively. Since $\boldsymbol{N}_{\boldsymbol{\sigma}}$ and $\boldsymbol{N}_{\boldsymbol{\sigma}} \times \dot{\boldsymbol{\gamma}}$ are perpendicular unit vectors we get

$$\|\ddot{\boldsymbol{\gamma}}\|^2 = \kappa_n^2 + \kappa_g^2$$

Hence, the curvature $\kappa = \|\ddot{\boldsymbol{\gamma}} \times \dot{\boldsymbol{\gamma}}\| / \|\dot{\boldsymbol{\gamma}}\|^3 \Rightarrow \|\ddot{\boldsymbol{\gamma}}\|$ of $\boldsymbol{\gamma}$ is given by $\kappa^2 = \kappa_n^2 + \kappa_g^2$

Since geodesic curvature is zero everywhere in the base curve, we observe $\kappa^2 = \kappa_n^2 = \langle \boldsymbol{\alpha}'', \boldsymbol{N_\sigma} \rangle^2$, and

$$0 = \kappa_g = \langle \boldsymbol{\alpha}'', \boldsymbol{N_\sigma} \times \boldsymbol{\alpha}' \rangle$$

Since the standard unit normal $\boldsymbol{N_\sigma}$ of the surface patch is defined by $\boldsymbol{N_\sigma} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|}$, we have

$$\langle \boldsymbol{\alpha}'', (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) \times \boldsymbol{\alpha}' \rangle = 0$$

The Frenet-Serret Equation of the Möbius Strip

Our main interest in this title will be to research, how we can express a curve $\boldsymbol{\gamma}(s)$ with non-vanishing curvature in the ruled surface $\boldsymbol{\sigma}(u, v)$ for Möbius strips with the Frenet-Serret parameter $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$, such that there exists one unique flat ruled surface, i.e.

$$\boldsymbol{\gamma}(s) = f(s) \cdot \boldsymbol{t}(s) + g(s) \cdot \boldsymbol{n}(s) + h(s) \cdot \boldsymbol{b}(s)$$

The main idea here is to use the curve $\boldsymbol{\gamma}$ as rulings $\boldsymbol{w}$ in the ruled surface $\boldsymbol{\sigma}$. To able to treat the hypothesis we use the following definition.

Definition 6

Let the Frenet-Serret parameter $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ along a regular unit-speed curve $\boldsymbol{\gamma}: I \to \mathbb{R}^3$ satisfy $\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \neq \mathbf{0}$ and the cross product relations

$$\boldsymbol{t} = \boldsymbol{n} \times \boldsymbol{b}, \qquad \boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{t}, \qquad \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$$

Where $\boldsymbol{t} = \dot{\boldsymbol{\gamma}} / \|\dot{\boldsymbol{\gamma}}\|$ is the tangent vector, $\boldsymbol{b} = \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} / \|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|$ is the binormal vector of $\boldsymbol{\gamma}$ and $\boldsymbol{n}$ is the unit principal normal of $\boldsymbol{\gamma}$.

These parameters give the Fernet-Serret equations

$$\begin{aligned} \dot{\boldsymbol{t}} &= & \kappa \boldsymbol{n} & \\ \dot{\boldsymbol{n}} &= -\kappa \boldsymbol{t} & + & \tau \boldsymbol{b} \\ \dot{\boldsymbol{b}} &= & -\tau \boldsymbol{n} & \end{aligned}$$

Where $\kappa = \|\ddot{\boldsymbol{\gamma}} \times \dot{\boldsymbol{\gamma}}\| / \|\dot{\boldsymbol{\gamma}}\|^3 > 0$ is the curvature and τ is the torsion of $\boldsymbol{\gamma}$ is $\tau = \langle \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}, \dddot{\boldsymbol{\gamma}} \rangle / \|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|^2$.

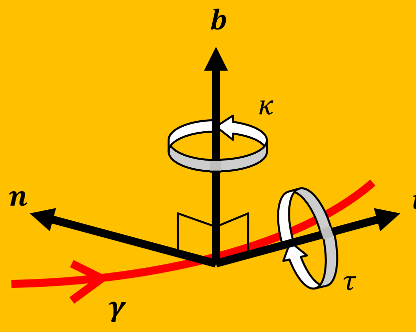


Figure 8 The influence of the curvature and the torsion.

A given curve with non-vanishing curvature there exists one unique flat ruled surface on which this curve is a geodesic.

$$\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \neq \mathbf{0} \Leftrightarrow \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} = \frac{\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}}{\|\ddot{\boldsymbol{\gamma}}\|} \neq \mathbf{0} \Leftrightarrow \dot{\boldsymbol{\gamma}} \text{ and } \ddot{\boldsymbol{\gamma}} \text{ are both nonzero}$$

The rectifying developable has the same normal vector as the principle normal vector of the center curve along the center curve. In the case of the cylindrical ruled surface the center curve is the base curve.

Let the ruled surface be defined by $\boldsymbol{\sigma}(u,v) = \boldsymbol{\alpha}(u) + v \cdot \boldsymbol{w}(u)$ with $\|\boldsymbol{\alpha}'(u)\| = 1$ and $\|\boldsymbol{w}(u)\| = 1$, so

$$\begin{aligned}\boldsymbol{\sigma}_u &= \boldsymbol{\alpha}' + v \cdot \boldsymbol{w}',\\ \boldsymbol{\sigma}_v &= \boldsymbol{w}.\end{aligned}$$

Put Fernet-Serret in the expression of $\boldsymbol{w}$ we get:

$$\boldsymbol{w} = f \cdot \boldsymbol{t} + g \cdot \boldsymbol{n} + h \cdot \boldsymbol{b}$$

where $\boldsymbol{t}(u) = \boldsymbol{\alpha}'(u)$ and arbitrary functions $f = f(u)$, $g = g(u)$ and $h = h(u)$. The expression gives

$$\begin{aligned}\boldsymbol{w}' &= f'\boldsymbol{t} + f\dot{\boldsymbol{t}} + g'\boldsymbol{n} + g\dot{\boldsymbol{n}} + h'\boldsymbol{b} + h\dot{\boldsymbol{b}}\\ &= f'\boldsymbol{t} + f\kappa\boldsymbol{n} + g'\boldsymbol{n} + g(-\kappa\boldsymbol{t} + \tau\boldsymbol{b}) + h'\boldsymbol{b} - h\tau\boldsymbol{n}\\ &= f'\boldsymbol{t} + f\kappa\boldsymbol{n} + g'\boldsymbol{n} - g\kappa\boldsymbol{t} + g\tau\boldsymbol{b} + h'\boldsymbol{b} - h\tau\boldsymbol{n}\\ &= (f' - g\kappa)\boldsymbol{t} + (f\kappa + g' - h\tau)\boldsymbol{n} + (g\tau + h')\boldsymbol{b}\end{aligned}$$

Our ruled surface has the Gaussian curvature $K = 0$ everywhere, therefore

$$\begin{aligned}\langle \boldsymbol{\alpha}' \times \boldsymbol{w}, \boldsymbol{w}' \rangle &= 0\\ \langle \boldsymbol{t} \times (f\boldsymbol{t} + g\boldsymbol{n} + h\boldsymbol{b}), (f' - g\kappa)\boldsymbol{t} + (f\kappa + g' - h\tau)\boldsymbol{n} + (g\tau + h')\boldsymbol{b} \rangle &= 0\\ \langle g \cdot (\boldsymbol{t} \times \boldsymbol{n}) + h \cdot (\boldsymbol{t} \times \boldsymbol{b}), (f' - g\kappa)\boldsymbol{t} + (f\kappa + g' - h\tau)\boldsymbol{n} + (g\tau + h')\boldsymbol{b} \rangle &= 0\\ \langle g\boldsymbol{b} - h\boldsymbol{n}, (f' - g\kappa)\boldsymbol{t} + (f\kappa + g' - h\tau)\boldsymbol{n} + (g\tau + h')\boldsymbol{b} \rangle &= 0\end{aligned}$$

We know that $\langle \boldsymbol{b}, \boldsymbol{t} \rangle = \langle \boldsymbol{t}, \boldsymbol{b} \rangle = 0, \langle \boldsymbol{b}, \boldsymbol{n} \rangle = \langle \boldsymbol{n}, \boldsymbol{b} \rangle = 0, \langle \boldsymbol{n}, \boldsymbol{t} \rangle = \langle \boldsymbol{t}, \boldsymbol{n} \rangle = 0$, then

$$g \cdot (g\tau + h')\langle \boldsymbol{b}, \boldsymbol{b} \rangle - h \cdot (f\kappa + g' - h\tau)\langle \boldsymbol{n}, \boldsymbol{n} \rangle = 0$$

$\boldsymbol{t}$ and $\boldsymbol{n}$ are unit vector for all u, so we have a different equation in the system

$$g \cdot (g\tau + h') - h \cdot (f\kappa + g' - h\tau) = 0$$

Equation 1

$$g \cdot (g\tau + h') = h \cdot (f\kappa + g' - h\tau)$$

The geodesic curvature is zero everywhere, so the next condition is

$$\langle \boldsymbol{\alpha}'', (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) \times \boldsymbol{\alpha}' \rangle = \langle \dot{\boldsymbol{t}}, (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) \times \boldsymbol{t} \rangle = 0$$

The first equation in the Fernet-Serret equations is $\dot{\boldsymbol{t}} = \kappa\boldsymbol{n}$, therefore

$$\langle \boldsymbol{n}, (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) \times \boldsymbol{t} \rangle = 0$$

Where

$$\begin{aligned}\boldsymbol{\sigma}_u &= \boldsymbol{t} + v \cdot \big((f' - g\kappa)\boldsymbol{t} + (f\kappa + g' - h\tau)\boldsymbol{n} + (g\tau + h')\boldsymbol{b}\big),\\ \boldsymbol{\sigma}_v &= f \cdot \boldsymbol{t} + g \cdot \boldsymbol{n} + h \cdot \boldsymbol{b}.\end{aligned}$$

Hence

$$
\begin{aligned}
\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= \Big(\boldsymbol{t} + v \cdot \big((f' - g\kappa)\boldsymbol{t} + (f\kappa + g' - h\tau)\boldsymbol{n} + (g\tau + h')\boldsymbol{b}\big)\Big) \times (f \cdot \boldsymbol{t} + g \cdot \boldsymbol{n} + h \cdot \boldsymbol{b}) \\
&= \boldsymbol{t} \times (f \cdot \boldsymbol{t} + g \cdot \boldsymbol{n} + h \cdot \boldsymbol{b}) + v \cdot \big((f' - g\kappa)\boldsymbol{t} + (f\kappa + g' - h\tau)\boldsymbol{n} + (g\tau + h')\boldsymbol{b}\big) \\
&\qquad \times (f \cdot \boldsymbol{t} + g \cdot \boldsymbol{n} + h \cdot \boldsymbol{b}) \\
&= g\boldsymbol{t} \times \boldsymbol{n} + h\boldsymbol{t} \times \boldsymbol{b} + v(f' - g\kappa)(g\boldsymbol{t} \times \boldsymbol{n} + h\boldsymbol{t} \times \boldsymbol{b}) + v(f\kappa + g' - h\tau)(f\boldsymbol{n} \times \boldsymbol{t} + h\boldsymbol{n} \times \boldsymbol{b}) \\
&\qquad + v(g\tau + h')(f\boldsymbol{b} \times \boldsymbol{t} + g\boldsymbol{b} \times \boldsymbol{n}) \\
&= g\boldsymbol{b} - h\boldsymbol{n} + v(f' - g\kappa)(g\boldsymbol{b} - h\boldsymbol{n}) + v(f\kappa + g' - h\tau)(-f\boldsymbol{b} + h\boldsymbol{t}) + v(g\tau + h')(f\boldsymbol{n} - g\boldsymbol{t}) \\
&= \big(1 + v(f' - g\kappa)\big)(g\boldsymbol{b} - h\boldsymbol{n}) + v(f\kappa + g' - h\tau)(-f\boldsymbol{b} + h\boldsymbol{t}) + v(g\tau + h')(f\boldsymbol{n} - g\boldsymbol{t})
\end{aligned}
$$

It leads us to

$$
\begin{aligned}
\langle \boldsymbol{n}, \Big(\big(1 + v(f' - g\kappa)\big)(g\boldsymbol{b} - h\boldsymbol{n}) + v(f\kappa + g' - h\tau)(-f\boldsymbol{b} + h\boldsymbol{t}) + v(g\tau + h')(f\boldsymbol{n} - g\boldsymbol{t})\Big) \times \boldsymbol{t} \rangle &= 0 \\
\langle \boldsymbol{n}, \big(1 + v(f' - g\kappa)\big)(g\boldsymbol{b} \times \boldsymbol{t} - h\boldsymbol{n} \times \boldsymbol{t}) + v(f\kappa + g' - h\tau)(-f\boldsymbol{b} \times \boldsymbol{t}) + v(g\tau + h')(f\boldsymbol{n} \times \boldsymbol{t}) \rangle &= 0 \\
\langle \boldsymbol{n}, \big(1 + v(f' - g\kappa)\big)(g\boldsymbol{n} + h\boldsymbol{b}) - v(f\kappa + g' - h\tau)f\boldsymbol{n} - v(g\tau + h')f\boldsymbol{b} \rangle &= 0 \\
\Big(\big(1 + v(f' - g\kappa)\big)g - v(f\kappa + g' - h\tau)f\Big)\langle \boldsymbol{n}, \boldsymbol{n} \rangle &= 0 \\
\big(1 + v(f' - g\kappa)\big)g - v(f\kappa + g' - h\tau)f &= 0
\end{aligned}
$$

Equation 2

$$\big(1 + v(f' - g\kappa)\big)g = v(f\kappa + g' - h\tau)f$$

The two equations are

Equation 1:

$$g \cdot (g\tau + h') = h \cdot (f\kappa + g' - h\tau)$$

Equation 2:

$$\big(1 + v(f' - g\kappa)\big)g = v(f\kappa + g' - h\tau)f$$

The geodesic condition is satisfied only if the curve is the base curve, i.e. $v = 0$. Therefore Equation 2 gives $g = 0$ everywhere, hence also $g' = 0$ and

Equation 1*:

$$0 = h \cdot (f\kappa - h\tau)$$

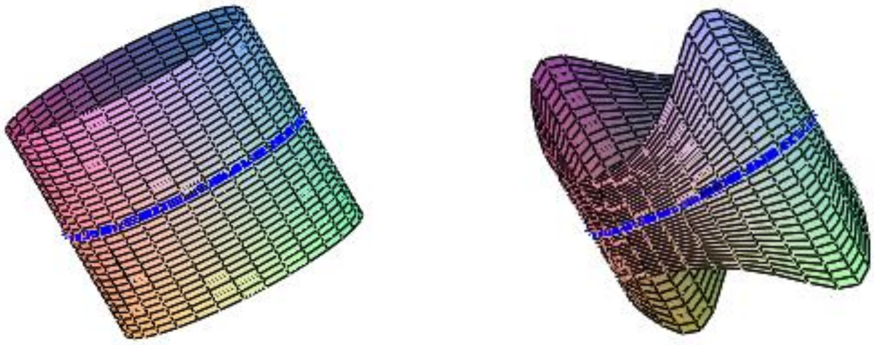
We are not interested in the obvious solution to the system as $h = 0$, then we assume that $h \neq 0$, hence Equation 3* gives

$$
\begin{aligned}
f\kappa - h\tau &= 0 \\
f\kappa &= h\tau
\end{aligned}
$$

There exist many solutions to the expression, but we can choose $f = \tau$ and $h = \kappa$ then we finally get

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\alpha}(u) + v\boldsymbol{w}(u) \text{ with } \boldsymbol{w}(u) = \tau\boldsymbol{t} + \kappa\boldsymbol{b}.$$

If one instead of chooses the ruling $\boldsymbol{w}(u) = \frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}$, i.e. $f = \tau/\kappa$ for $h = 1$ the parameter v is the geodesic distance to the base curve. There exist examples that can illustrate the situation like the figure below.



$\boldsymbol{\sigma}(u,v) = \boldsymbol{\alpha}(u) + v\left(\frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}\right)$ $\qquad$ $\boldsymbol{\sigma}(u,v) = \boldsymbol{\alpha}(u) + v(\tau\boldsymbol{t} + \kappa\boldsymbol{b})$

Figure 9 A surface with the base curve $\boldsymbol{\alpha}(u) = (\cos(u), \sin(u), \sin(u))$

If we want to have $\boldsymbol{w}$ to be a unit vector, then

$$\begin{aligned} 1 &= \langle \boldsymbol{w}, \boldsymbol{w} \rangle \\ &= \langle f \cdot \boldsymbol{t} + g \cdot \boldsymbol{n} + h \cdot \boldsymbol{b}, f \cdot \boldsymbol{t} + g \cdot \boldsymbol{n} + h \cdot \boldsymbol{b} \rangle \\ &= f^2 \cdot \langle \boldsymbol{t}, \boldsymbol{t} \rangle + g^2 \cdot \langle \boldsymbol{n}, \boldsymbol{n} \rangle + h^2 \cdot \langle \boldsymbol{b}, \boldsymbol{b} \rangle \\ &= f^2 + g^2 + h^2 \end{aligned}$$

It means that for $\boldsymbol{w} = \frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}$ we have

$$\left(\frac{\tau}{\kappa}\right)^2 + 1^2 = 1 \Leftrightarrow \tau = 0$$

The result tells us that $\boldsymbol{w} = \frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}$ can only be a unit vector, if the torsion of the surface is zero along the geodesic, i.e. the base curve. Consequently is the base curve is planar.

Now we make use of the following examples to prove that the parameter v in the first surface in the figure above is the geodesic distance to the base curve, while the parameter v in the second surface doesn't that.

Example 6

We define a curve $\boldsymbol{\gamma}$ on a surface $\boldsymbol{\sigma}$ such that $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}\big(u(t), v(t)\big)$, then we find

$$\begin{aligned}
\dot{\boldsymbol{\gamma}} &= \boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v} \\
\ddot{\boldsymbol{\gamma}} &= (\boldsymbol{\sigma}_{uu}\dot{u} + \boldsymbol{\sigma}_{uv}\dot{v})\dot{u} + \boldsymbol{\sigma}_u \ddot{u} + (\boldsymbol{\sigma}_{uv}\dot{u} + \boldsymbol{\sigma}_{vv}\dot{v})\dot{v} + \boldsymbol{\sigma}_v \ddot{v} \\
&= \boldsymbol{\sigma}_{uu}\dot{u}^2 + 2\boldsymbol{\sigma}_{uv}\dot{v}\dot{u} + \boldsymbol{\sigma}_{vv}\dot{v}^2 + \boldsymbol{\sigma}_u \ddot{u} + \boldsymbol{\sigma}_v \ddot{v}
\end{aligned}$$

A curve is geodesic, if the geodesic curvature is zero

$$\begin{aligned}
\kappa_g &= \langle \ddot{\boldsymbol{\gamma}}, \boldsymbol{N}_{\boldsymbol{\sigma}} \times \dot{\boldsymbol{\gamma}} \rangle \\
&= \langle \boldsymbol{\sigma}_{uu}\dot{u}^2 + 2\boldsymbol{\sigma}_{uv}\dot{v}\dot{u} + \boldsymbol{\sigma}_{vv}\dot{v}^2 + \boldsymbol{\sigma}_u \ddot{u} + \boldsymbol{\sigma}_v \ddot{v}, \boldsymbol{N}_{\boldsymbol{\sigma}} \times (\boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}) \rangle
\end{aligned}$$

v is a ruling, if $v = constant$, i.e. $\dot{v} = 0$, hence also $\ddot{v} = 0$.

$$\kappa_g = \langle \boldsymbol{\sigma}_{uu}\dot{u}^2 + \boldsymbol{\sigma}_u \ddot{u}, \boldsymbol{N}_{\boldsymbol{\sigma}} \times \boldsymbol{\sigma}_u \dot{u} \rangle$$

So for a surface defined by $\boldsymbol{\sigma}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{w}(u)$ with $\boldsymbol{w}(u) = \frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}$, we have

$$\begin{aligned}
\boldsymbol{\alpha}_u &= \boldsymbol{t} \\
\boldsymbol{\alpha}_{uu} &= \boldsymbol{t}_u \\
&= \kappa \boldsymbol{n} \\
\boldsymbol{w}_u &= \left(\frac{\tau_u \kappa - \tau \kappa_u}{\kappa^2}\right)\boldsymbol{t} + \frac{\tau}{\kappa}\boldsymbol{t}_u + \boldsymbol{b}_u \\
&= \left(\frac{\tau_u \kappa - \tau \kappa_u}{\kappa^2}\right)\boldsymbol{t} + \tau \boldsymbol{n} - \tau \boldsymbol{n} \\
&= \left(\frac{\tau_u \kappa - \tau \kappa_u}{\kappa^2}\right)\boldsymbol{t} \\
\boldsymbol{w}_{uu} &= \left(\frac{\tau_{uu}\kappa^2 - \tau\kappa\kappa_{uu} + 2(\tau\kappa_u - \tau_u\kappa)\kappa_u}{\kappa^3}\right)\boldsymbol{t} + \left(\frac{\tau_u \kappa - \tau \kappa_u}{\kappa^2}\right)\boldsymbol{t}_u \\
&= \left(\frac{\tau_{uu}\kappa^2 - \tau\kappa\kappa_{uu} + 2(\tau\kappa_u - \tau_u\kappa)\kappa_u}{\kappa^3}\right)\boldsymbol{t} + \kappa\left(\frac{\tau_u \kappa - \tau \kappa_u}{\kappa^2}\right)\boldsymbol{n}
\end{aligned}$$

Which lead us to

$$\begin{aligned}
\boldsymbol{\sigma}_v &= \boldsymbol{w} \\
&= \frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b} \\
\boldsymbol{\sigma}_u &= \boldsymbol{\alpha}_u + v\boldsymbol{w}_u \\
&= \boldsymbol{t} + v\left(\frac{\tau_u \kappa - \tau \kappa_u}{\kappa^2}\right)\boldsymbol{t} \\
&= \left(1 + v\left(\frac{\tau_u \kappa - \tau \kappa_u}{\kappa^2}\right)\right)\boldsymbol{t} \\
\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= \left(\frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}\right) \times \left(1 + v\left(\frac{\tau_u \kappa - \tau \kappa_u}{\kappa^2}\right)\right)\boldsymbol{t} \\
&= \left(1 + v\left(\frac{\tau_u \kappa - \tau \kappa_u}{\kappa^2}\right)\right)\boldsymbol{b} \times \boldsymbol{t} \\
&= -\left(1 + v\left(\frac{\tau_u \kappa - \tau \kappa_u}{\kappa^2}\right)\right)\boldsymbol{n}
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{N}_\sigma &= \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} \\
&= -\frac{\left(1 + v\left(\frac{\tau_u\kappa - \tau\kappa_u}{\kappa^2}\right)\right)\boldsymbol{n}}{\left\|-\left(1 + v\left(\frac{\tau_u\kappa - \tau\kappa_u}{\kappa^2}\right)\right)\boldsymbol{n}\right\|} \\
&= -\boldsymbol{n} \\
\boldsymbol{\sigma}_{uu} &= \boldsymbol{\alpha}_{uu} + v\boldsymbol{w}_{uu} \\
&= \kappa\boldsymbol{n} + v\left(\left(\frac{\tau_{uu}\kappa^2 - \tau\kappa\kappa_{uu} + 2(\tau\kappa_u - \tau_u\kappa)\kappa_u}{\kappa^3}\right)\boldsymbol{t} + \kappa\left(\frac{\tau_u\kappa - \tau\kappa_u}{\kappa^2}\right)\boldsymbol{n}\right) \\
&= \left(1 + v\left(\frac{\tau_u\kappa - \tau\kappa_u}{\kappa^2}\right)\right)\kappa\boldsymbol{n} + v\left(\frac{\tau_{uu}\kappa^2 - \tau\kappa\kappa_{uu} + 2(\tau\kappa_u - \tau_u\kappa)\kappa_u}{\kappa^3}\right)\boldsymbol{t} \\
\boldsymbol{\sigma}_{vv} &= \boldsymbol{0} \\
\boldsymbol{\sigma}_{uv} &= \boldsymbol{w}_u \\
&= \left(\frac{\tau_u\kappa - \tau\kappa_u}{\kappa^2}\right)\boldsymbol{t}
\end{aligned}
$$

We determine the following expression

$$
\begin{aligned}
\boldsymbol{N}_\sigma \times \boldsymbol{\sigma}_u\dot{u} &= -\boldsymbol{n} \times \left(1 + v\left(\frac{\tau_u\kappa - \tau\kappa_u}{\kappa^2}\right)\right)\boldsymbol{t}\dot{u} \\
&= -\dot{u}\left(1 + v\left(\frac{\tau_u\kappa - \tau\kappa_u}{\kappa^2}\right)\right)^2 \boldsymbol{n} \times \boldsymbol{t} \\
&= \dot{u}\left(1 + v\left(\frac{\tau_u\kappa - \tau\kappa_u}{\kappa^2}\right)\right)^2 \boldsymbol{b}
\end{aligned}
$$

Therefore we find

$$\kappa_g = \langle \boldsymbol{\sigma}_{uu}\dot{u}^2 + \boldsymbol{\sigma}_u\ddot{u}, \boldsymbol{N}_\sigma \times \boldsymbol{\sigma}_u\dot{u}\rangle = 0$$

The result shows that the parameter v in $\boldsymbol{\sigma}(u,v) = \boldsymbol{\alpha}(u) + v\left(\frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}\right)$ is the geodesic distance to the base curve.

Example 7

Now for a surface defined by $\boldsymbol{\sigma}(u,v) = \boldsymbol{\alpha}(u) + v\boldsymbol{w}(u)$ with $\boldsymbol{w}(u) = \tau\boldsymbol{t} + \kappa\boldsymbol{b}$, we have

$$
\begin{aligned}
\boldsymbol{\alpha}_u &= \boldsymbol{t} \\
\boldsymbol{\alpha}_{uu} &= \boldsymbol{t}_u \\
&= \kappa\boldsymbol{n} \\
\boldsymbol{w}_u &= \tau_u\boldsymbol{t} + \tau\boldsymbol{t}_u + \kappa_u\boldsymbol{b} + \kappa\boldsymbol{b}_u \\
&= \tau_u\boldsymbol{t} + \tau\kappa\boldsymbol{n} + \kappa_u\boldsymbol{b} - \tau\kappa\boldsymbol{n} \\
&= \tau_u\boldsymbol{t} + \kappa_u\boldsymbol{b} \\
\boldsymbol{w}_{uu} &= \tau_{uu}\boldsymbol{t} + \tau_u\boldsymbol{t}_u + \kappa_{uu}\boldsymbol{b} + \kappa_u\boldsymbol{b}_u \\
&= \tau_{uu}\boldsymbol{t} + \tau_u\kappa\boldsymbol{n} + \kappa_{uu}\boldsymbol{b} - \tau\kappa_u\boldsymbol{n} \\
&= \tau_{uu}\boldsymbol{t} + (\tau_u\kappa - \tau\kappa_u)\boldsymbol{n} + \kappa_{uu}\boldsymbol{b}
\end{aligned}
$$

Which lead us to

$$
\begin{aligned}
\boldsymbol{\sigma}_v &= \boldsymbol{w} \\
&= \tau\boldsymbol{t} + \kappa\boldsymbol{b} \\
\boldsymbol{\sigma}_u &= \boldsymbol{\alpha}_u + v\boldsymbol{w}_u \\
&= \boldsymbol{t} + v(\tau_u\boldsymbol{t} + \kappa_u\boldsymbol{b}) \\
&= (1 + v\tau_u)\boldsymbol{t} + v\kappa_u\boldsymbol{b} \\
\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= \big((1 + v\tau_u)\boldsymbol{t} + v\kappa_u\boldsymbol{b}\big) \times (\tau\boldsymbol{t} + \kappa\boldsymbol{b}) \\
&= (1 + v\tau_u)\kappa\boldsymbol{t} \times \boldsymbol{b} + v\tau\kappa_u\boldsymbol{b} \times \boldsymbol{t} \\
&= (1 + v\tau_u)\kappa\boldsymbol{n} - v\tau\kappa_u\boldsymbol{n} \\
&= \big(\kappa + v(\tau_u\kappa - \tau\kappa_u)\big)\boldsymbol{n} \\
\boldsymbol{N}_\sigma &= \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} \\
&= \frac{\big(\kappa + v(\tau_u\kappa - \tau\kappa_u)\big)\boldsymbol{n}}{\big\|\big(\kappa + v(\tau_u\kappa - \tau\kappa_u)\big)\boldsymbol{n}\big\|} \\
&= \boldsymbol{n} \\
\boldsymbol{\sigma}_{uu} &= \boldsymbol{\alpha}_{uu} + v\boldsymbol{w}_{uu} \\
&= \kappa\boldsymbol{n} + v(\tau_{uu}\boldsymbol{t} + (\tau_u\kappa - \tau\kappa_u)\boldsymbol{n} + \kappa_{uu}\boldsymbol{b}) \\
&= v\tau_{uu}\boldsymbol{t} + \big(\kappa + v(\tau_u\kappa - \tau\kappa_u)\big)\boldsymbol{n} + v\kappa_{uu}\boldsymbol{b}
\end{aligned}
$$

We determine the following two expressions

$$
\begin{aligned}
\boldsymbol{\sigma}_{uu}\dot{u}^2 + \boldsymbol{\sigma}_u\ddot{u} &= \big(v\tau_{uu}\boldsymbol{t} + \big(\kappa + v(\tau_u\kappa - \tau\kappa_u)\big)\boldsymbol{n} + v\kappa_{uu}\boldsymbol{b}\big)\dot{u}^2 + \big((1 + v\tau_u)\boldsymbol{t} + v\kappa_u\boldsymbol{b}\big)\ddot{u} \\
&= (v\tau_{uu}\dot{u}^2 + (1 + v\tau_u)\ddot{u})\boldsymbol{t} + \big(\kappa + v(\tau_u\kappa - \tau\kappa_u)\big)\dot{u}^2\boldsymbol{n} + (v\kappa_{uu}\dot{u}^2 + v\kappa_u\ddot{u})\boldsymbol{b}
\end{aligned}
$$

And

$$
\begin{aligned}
\boldsymbol{N}_\sigma \times \boldsymbol{\sigma}_u\dot{u} &= \boldsymbol{n} \times \big((1 + v\tau_u)\boldsymbol{t} + v\kappa_u\boldsymbol{b}\big)\dot{u} \\
&= \dot{u}\big((1 + v\tau_u)\boldsymbol{n} \times \boldsymbol{t} + v\kappa_u\boldsymbol{n} \times \boldsymbol{b}\big) \\
&= \dot{u}(-(1 + v\tau_u)\boldsymbol{b} + v\kappa_u\boldsymbol{t})
\end{aligned}
$$

Therefore we find

$$
\begin{aligned}
\kappa_g &= \langle \boldsymbol{\sigma}_{uu}\dot{u}^2 + \boldsymbol{\sigma}_u\ddot{u}, \boldsymbol{N}_\sigma \times \boldsymbol{\sigma}_u\dot{u} \rangle \\
&= \dot{u}\big((v\tau_{uu}\dot{u}^2 + (1 + v\tau_u)\ddot{u})v\kappa_u\langle \boldsymbol{t}, \boldsymbol{t} \rangle - (v\kappa_{uu}\dot{u}^2 + v\kappa_u\ddot{u})(1 + v\tau_u)\langle \boldsymbol{b}, \boldsymbol{b} \rangle\big) \\
&= \dot{u}\big((v\tau_{uu}\dot{u}^2 + (1 + v\tau_u)\ddot{u})v\kappa_u - (v\kappa_{uu}\dot{u}^2 + v\kappa_u\ddot{u})(1 + v\tau_u)\big) \\
&= \dot{u}\big(v\tau_{uu}\dot{u}^2v\kappa_u + (1 + v\tau_u)\ddot{u}v\kappa_u - v\kappa_{uu}\dot{u}^2(1 + v\tau_u) - v\kappa_u\ddot{u}(1 + v\tau_u)\big) \\
&= \dot{u}^3v\big(\tau_{uu}v\kappa_u - \kappa_{uu}(1 + v\tau_u)\big)
\end{aligned}
$$

Here we can think that we can't guarantee that the geodesic curvature is zero along the ruling v, thus it is not sure that v is the geodesic distance to the base curve.

For rectifying developable ruled surface $\dot{\boldsymbol{w}} \times \ddot{\boldsymbol{w}} \neq \boldsymbol{0}$ with $\boldsymbol{w}(u) = \frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}$ we check

$$
\begin{aligned}
\dot{\boldsymbol{w}} \times \ddot{\boldsymbol{w}} &= \left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\boldsymbol{t} \times \left(\left(\frac{\ddot{\tau}\kappa^2 - \tau\kappa\ddot{\kappa} + 2(\tau\dot{\kappa} - \dot{\tau}\kappa)\dot{\kappa}}{\kappa^3}\right)\boldsymbol{t} + \kappa\left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\boldsymbol{n}\right) \\
&= \kappa\left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)^2 \boldsymbol{t} \times \boldsymbol{n} \\
&= \kappa\left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)^2 \boldsymbol{b}
\end{aligned}
$$

We can conclude that $\dot{\boldsymbol{w}} \times \ddot{\boldsymbol{w}} \neq \mathbf{0}$, if and only if $\kappa \neq 0, \frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2} \neq 0$ or $\boldsymbol{b} \neq \mathbf{0}$. The second condition can reformulate such that

$$\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2} \neq 0$$
$$\frac{d}{du}\left(\frac{\tau}{\kappa}\right) \neq 0$$
$$\frac{\tau}{\kappa} \neq C$$

With an arbitrary constant C.

The Bending Energy in the Möbius Strip

In this section we calculate, how much a given Möbius strip is bending. There exists a quantitative calculate of how much a given surface deviates from a round sphere, and it is called Willmore energy or is also called the bending energy. The energy of a smooth closed surface $\boldsymbol{\sigma}$ is defined by

$$\varepsilon = \int_{\boldsymbol{\sigma}} (H^2 - K)dA$$

Where H is the mean curvature, K is the Gaussian curvature, and dA is the area form of $\boldsymbol{\sigma}$. The formula for the element of area of $\boldsymbol{\sigma}$ is a part of the surface integral. To define a surface integral, we have taken a surface $\boldsymbol{\sigma}$, where u and v varies over a region S in uv-plane. Since we assume the surface $\boldsymbol{\sigma}$ is smooth, we get $\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|$, which is the area of the parallelogram with sides $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ by the concept of cross product. Hence

$$\begin{aligned} dA &= \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| dudv \\ &= \sqrt{\langle \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \rangle} dudv \\ &= \sqrt{\langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_u \rangle \langle \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_v \rangle - \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle^2} dudv \\ &= \sqrt{EG - F^2} dvdu \end{aligned}$$

Therefore the energy is

$$\varepsilon = \int\int_S (H^2 - K)\sqrt{EG - F^2}\, dv\, du$$

The bending energy is always greater or equal to zero.

Let a flat ruled surface in $\mathbb{R}^3$ be defined by

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{w}(u)$$

With the ruling $\boldsymbol{w}(u) = \cos\left(\frac{N\cdot\pi}{L}\cdot u\right)\left(\frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}\right) + \sin\left(\frac{N\cdot\pi}{L}\cdot u\right)\boldsymbol{n}$ for $N = 0$, i.e. $\boldsymbol{w}(u) = \frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}$. The parameters u and v in the surface are both geodesics, so we can define the region as a rectangle, i.e. $(u, v) \in S \equiv [0, L] \times \left[-\frac{d}{2}, \frac{d}{2}\right]$.

The first fundamental form of the surface is

$$\begin{aligned} E &= \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_u \rangle \\ &= \left(1 + v \cdot \left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\right)^2 \langle \boldsymbol{t}, \boldsymbol{t} \rangle \\ &= \left(1 + v \cdot \left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\right)^2 \\ F &= \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v \rangle \\ &= \left\langle \left(1 + v \cdot \left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\right)\boldsymbol{t}, \frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b} \right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{\tau}{\kappa}\left(1 + v \cdot \left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\right)\langle \boldsymbol{t}, \boldsymbol{t}\rangle \\
&= \frac{\tau}{\kappa}\left(1 + v \cdot \left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\right) \\
&= \frac{\tau}{\kappa}\sqrt{E} \\
G &= \langle \boldsymbol{\sigma}_v, \boldsymbol{\sigma}_v \rangle \\
&= \langle \frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}, \frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}\rangle \\
&= \left(\frac{\tau}{\kappa}\right)^2 \langle \boldsymbol{t}, \boldsymbol{t}\rangle + \langle \boldsymbol{b}, \boldsymbol{b}\rangle \\
&= \left(\frac{\tau}{\kappa}\right)^2 + 1
\end{aligned}
$$

and the second fundamental form

$$
\begin{aligned}
L &= \langle \boldsymbol{N}_{\boldsymbol{\sigma}}, \boldsymbol{\sigma}_{uu} \rangle \\
&= \langle -\boldsymbol{n}, \left(1 + v\left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\right)\kappa\boldsymbol{n} + v\left(\frac{\ddot{\tau}\kappa^2 - \tau\kappa\ddot{\kappa} + 2(\tau\dot{\kappa} - \dot{\tau}\kappa)\dot{\kappa}}{\kappa^3}\right)\boldsymbol{t}\rangle \\
&= -\kappa\left(1 + v\left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\right)\langle \boldsymbol{n}, \boldsymbol{n}\rangle \\
&= -\kappa\left(1 + v\left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\right) \\
&= -\kappa\sqrt{E} \\
N &= \langle \boldsymbol{N}_{\boldsymbol{\sigma}}, \boldsymbol{\sigma}_{vv} \rangle \\
&= 0 \\
M &= \langle \boldsymbol{N}_{\boldsymbol{\sigma}}, \boldsymbol{\sigma}_{uv} \rangle \\
&= \langle -\boldsymbol{n}, \left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)\boldsymbol{t}\rangle \\
&= 0
\end{aligned}
$$

Which leads us to the two expressions

$$
\begin{aligned}
EG - F^2 &= E\left(\left(\frac{\tau}{\kappa}\right)^2 + 1\right) - \left(\frac{\tau}{\kappa}\sqrt{E}\right)^2 = E \\
LG - 2MF + NE &= -\kappa\sqrt{E}G
\end{aligned}
$$

Therefore we get the mean curvature:

$$
\begin{aligned}
H &= \frac{LG - 2MF + NE}{2(EG - F^2)} \\
&= \frac{-\kappa\sqrt{E}G}{2E} \\
&= -\frac{\kappa G}{2\sqrt{E}}
\end{aligned}
$$

And not surprising we get the Gaussian curvature $K = 0$ everywhere. So the bending energy becomes

$$
\begin{aligned}
\varepsilon &= \int_0^L \int_{-d/2}^{d/2} H^2\sqrt{EG - F^2}\,dv\,du \\
&= \int_0^L \int_{-d/2}^{d/2} \left(-\frac{\kappa G}{2\sqrt{E}}\right)^2 \sqrt{E}\,dv\,du \\
&= \frac{1}{4}\int_0^L \kappa^2 \int_{-d/2}^{d/2} \frac{G^2}{\sqrt{E}}\,dv\,du \\
&= \frac{1}{4}\int_0^L \kappa^2 \int_{-d/2}^{d/2} \frac{\left(\left(\frac{\tau}{\kappa}\right)^2 + 1\right)^2}{1 + v\cdot\left(\frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\right)}\,dv\,du
\end{aligned}
$$

Let us assume a new function $\psi = \frac{1}{\dot{\tau}\kappa - \tau\dot{\kappa}}$, so we have

$$
\begin{aligned}
\varepsilon &= \frac{1}{4}\int_0^L \kappa^2\left(\left(\frac{\tau}{\kappa}\right)^2 + 1\right)^2 \int_{-d/2}^{d/2} \frac{1}{1 + \frac{v}{\kappa^2\psi}}\,dv\,du \\
&= \frac{1}{4}\int_0^L \kappa^2\left(\left(\frac{\tau}{\kappa}\right)^2 + 1\right)^2 \left[\kappa^2\psi \ln\left(1 + \frac{v}{\kappa^2\psi}\right)\right]_{v=-d/2}^{v=d/2} du \\
&= \frac{1}{4}\int_0^L \kappa^4\left(\left(\frac{\tau}{\kappa}\right)^2 + 1\right)^2 \psi\cdot\left[\ln\left(1 + \frac{d}{2\kappa^2\psi}\right) - \ln\left(1 - \frac{d}{2\kappa^2\psi}\right)\right] du \\
&= \frac{1}{4}\int_0^L (\tau^2 + \kappa^2)^2\psi\cdot\ln\left(\frac{1 + \frac{d}{2\kappa^2\psi}}{1 - \frac{d}{2\kappa^2\psi}}\right) du \\
&= \frac{1}{4}\int_0^L (\tau^2 + \kappa^2)^2\psi\cdot\ln\left(\frac{2\kappa^2\psi + d}{2\kappa^2\psi - d}\right) du
\end{aligned}
$$

The result shows the bending energy is zero along the base curve, $d = 0$, and is going numerically to infinity, if the value of d is going numerically to $2\kappa^2\psi$. To get a non complex number of the bending energy, we have to demand that the width of the strip is bounded by the interval $v \in \left[-\frac{d}{2}, \frac{d}{2}\right]$ with $0 \le d < 2\kappa^2\psi$.

The Minimal Bending Energy

There are many different Möbius strips, but we are interesting in strips that give minimal bending energy. How we can find these?

It is a method that can tell us, how we can draw a smooth centerline at every Möbius strip through a given number of points (called control points) without to use too much bending energy? The method use interpolation technique to analyze and correlate the knots efficiently. The method is called cubic spline curve. It allowed us to describe a complex smooth curve with relativity simplex piecewise cubic polynomial functions. From high school we know that the polynomials have the general form

$$p(x) = a + bx + cx^2 + dx^3 + \cdots$$

which the degree of a polynomial corresponds with the highest coefficient that is non-zero, e.g. if c is non-zero while the coefficients d and higher are all zero, the polynomial is of degree two. If d is the highest non-zero coefficient, the polynomial is called cubic. The degree three polynomials are most typically chosen to building a smooth curve, because it is the lowest degree polynomials that can support an inflection and polynomials with degree higher than three tend to be very sensitive to knots. The essential idea of the cubic spline can shortly describes by the following definition and is illustrated in figure 10.

Definition 7

Let $x_0 < x_1 < \cdots < x_n$ be given knots. A function s is said to be a cubic spline on the interval $[x_0, x_n]$, if s, s' and s'' are continuous in $[x_0, x_n]$, and s is a polynomial of degree three or less in each knot interval $[x_{i-1}, x_i]$ for $i = 1, \dots, n$.

It means that the spline s has n cubic polynomials,

$$s(x) = \begin{cases} s_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3, & x_0 \leq x \leq x_1 \\ s_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3, & x_1 \leq x \leq x_2 \\ \vdots & \vdots \\ s_n(x) = a_n + b_n x + c_n x^2 + d_n x^3, & x_{n-1} \leq x \leq x_n \end{cases}$$

Here we see that each s_i has four coefficients, so there are $4n$ coefficents to be determined, but the continuity requirements give $3(n-1)$ conditions,

$$\left.\begin{aligned} s_i(x_i) &= s_{i+1}(x_i) \\ s_i'(x_i) &= s_{i+1}'(x_i) \\ s_i''(x_i) &= s_{i+1}''(x_i) \end{aligned}\right\} \quad i = 1, \dots, n-1$$

So now we have $4n - 3(n-1) = n + 3$ degrees of freedom. We choose to require that the spline has to interpolate in the knots, $s(x_i) = f(x_i)$ for $i = 0,1, \dots, n$. The decision gives $n + 1$ conditions, which remains two conditions to us.

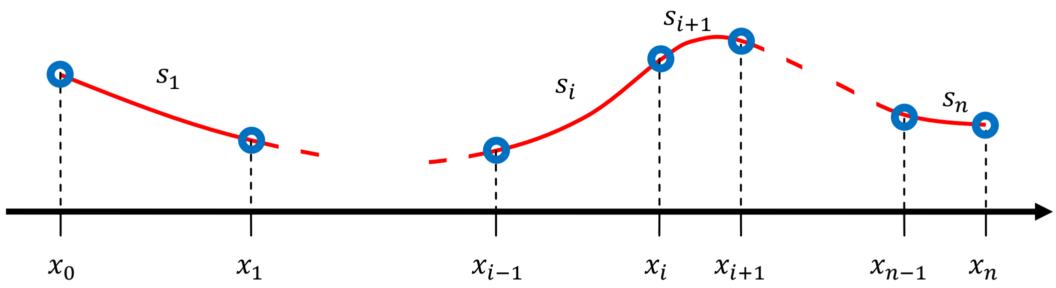


Figure 10 The cubic spline

The correct boundary conditions of the two remaining conditions will be:

$$s_1'(x_0) = f'(x_0)$$
$$s_n'(x_n) = f'(x_n)$$

But here we want to use the periodic boundary conditions, i.e. $s_1^{(k)}(x_0) = s_n^{(k)}(x_n)$ for $k = 1,2$. The periodic boundary conditions are used for interpolation of periodic function. We already have periodicity of the knot values of s itself from the conditions $s_1(x_0) = f(x_0) = f(x_n) = s_n(x_n)$, and the extra conditions ensure that the first and second derivative also are periodic.

Naturally there exist other conditions, e.g. natural spline, "Not-a-knot" and so on, but right now we have not need to concentrate on these, because we are only interested in producing a cubic spline of the centerline at the Möbius strip. The centerline is closed therefore can describe of a periodic function. How we use the spline in three dimensional will be explained later in this chapter.

If we prescribe that $s_i(x_{i-1}) = f_{i-1}$ and $s_i(x_i) = f_i$ for $i = 1,2,\dots,n$, we can expect that the interpolation property and the continuity of s are satisfied. The continuity of s' is satisfied if and only if $s_i'(x_{i-1}) = s_{i-1}'$ and $s_i'(x_i) = s_i'$ for $i = 1,2,\dots,n$. We don't know all slopes $\{s_j'\}$ in the knots, but to our lucky we can use Hermite interpolation with of degree three to determine the slopes $\{s_j'\}$. Suppose the polynomial

$$s_i(x) = a_i + b_i u + c_i u^2 + d_i u^3$$

Where $u = \frac{x - x_{i-1}}{h_i}$ with $h_i = x_i - x_{i-1}$. Note that $s_i(x_i) \Rightarrow u = 1$ and $s_{i+1}(x_i) \Rightarrow u = 0$.

If we try to derivate $s_i(x)$, we get

$$s_i'(x) = \frac{b_i + 2c_i u + 3d_i u^2}{h_i}, \qquad s_i''(x) = \frac{2c_i + 6d_i u}{h_i^2}$$

As said we need to satisfy the condition $s_i''(x_i) = s_{i+1}''(x_i)$ at the interior knots, which is equivalent to

$$\frac{2c_i + 6d_i}{h_i^2} = \frac{2c_{i+1}}{h_{i+1}^2}$$

For $i = 1,2,\dots,n-1$. From the book [21] we know that the Hermite interpolation gives

$$a_i = f_{i-1}$$
$$b_i = h_i s_{i-1}'$$
$$c_i = 3(f_i - f_{i-1}) - h_i(2s_{i-1}' + s_i')$$
$$d_i = 2(f_{i-1} - f_i) + h_i(s_{i-1}' + s_i')$$

We put expressions of c_i and d_i into the conditions, so we get

$$-6\frac{f_i - f_{i-1}}{h_i^2} + \frac{2s'_{i-1} + 4s'_i}{h_i} = 6\frac{f_{i+1} - f_i}{h_{i+1}^2} - \frac{4s'_i + 2s'_{i+1}}{h_{i+1}}$$

$$\frac{2s'_{i-1} + 4s'_i}{h_i} + \frac{4s'_i + 2s'_{i+1}}{h_{i+1}} = 6\left(\frac{f_{i+1} - f_i}{h_{i+1}^2} + \frac{f_i - f_{i-1}}{h_i^2}\right)$$

After we multiply the equation with $h_i h_{i+1}/2$, we have

$$h_{i+1}(s'_{i-1} + 2s'_i) + h_i(2s'_i + s'_{i+1}) = r_i$$
$$h_{i+1}s'_{i-1} + 2(h_i + h_{i+1})s'_i + h_i s'_{i+1} = r_i$$

for $i = 1,2, \dots, n-1$, where

$$r_i = 3\left(h_i\frac{f_{i+1} - f_i}{h_{i+1}} + h_{i+1}\frac{f_i - f_{i-1}}{h_i}\right)$$

Since this is a system of $n-1$ linear equations in the $n+1$ unknowns $\{s'_j\}_{j=0}^{n}$, we have as predicted two degree of freedom. A periodic spline must further satisfy the two equations $s'(x_0) = s'(x_n)$ and $s''(x_0) = s''(x_n)$ and . The left side of the first condition is

$$s'(x_0) = s'_1(x_0) = \frac{b_1}{h_1} = s'_0$$

While the right side is

$$s'(x_n) = s'_n(x_n) = \frac{b_n + 2c_n + 3d_n}{h_n} = s'_n$$

This condition leads to the simple expression $s'_0 - s'_n = r_0$ where $r_0 = 0$.

And the left side of the second condition is

$$s''(x_0) = s''_1(x_0) = \frac{2c_1}{h_1^2} = 6\frac{f_1 - f_0}{h_1^2} - \frac{4s'_0 + 2s'_1}{h_1}$$

And the right side

$$s''(x_n) = s''_n(x_n) = \frac{2c_n + 6d_n}{h_n^2} = 6\frac{f_n - f_{n-1}}{h_n^2} + \frac{2s'_{n-1} + 4s'_n}{h_n}$$

This condition leads to

$$6\frac{f_1 - f_0}{h_1^2} - \frac{4s'_0 + 2s'_1}{h_1} = -6\frac{f_n - f_{n-1}}{h_n^2} + \frac{2s'_{n-1} + 4s'_n}{h_n}$$
$$\frac{2s'_{n-1} + 4s'_n}{h_n} + \frac{4s'_0 + 2s'_1}{h_1} = 6\left(\frac{f_1 - f_0}{h_1^2} + \frac{f_n - f_{n-1}}{h_n^2}\right)$$

Now we multiply the equation with $h_1 h_n/2$, so we have

$$h_1(s'_{n-1} + 2s'_n) + h_n(2s'_0 + s'_1) = r_n$$
$$2h_n s'_0 + h_n s'_1 + h_1 s'_{n-1} + 2h_1 s'_n = r_n$$

Where $r_n = 3\left(h_n \frac{f_1 - f_0}{h_1} + h_1 \frac{f_n - f_{n-1}}{h_n}\right)$.

We can write the system in matrix form for the periodic spline

$$\begin{bmatrix} 1 & & & & & -1 & \\ h_2 & 2(h_1+h_2) & h_1 & & & & \\ & \ddots & \ddots & & \ddots & & \\ & & h_n & 2(h_{n-1}+h_n) & & h_{n-1} & \\ & 2h_n & h_n & & h_1 & & 2h_1 \end{bmatrix} \begin{bmatrix} s'_0 \\ s'_1 \\ \vdots \\ s'_{n-1} \\ s'_n \end{bmatrix} = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \\ r_n \end{bmatrix}$$

Remember a continuous function can be periodic if and only if $s(x_0) = s(x_n)$.

Now we can use the set $\{s'_j\}_{j=0}^{n}$ to solve the coefficients

$$a_i = f_{i-1}$$
$$b_i = h_i s'_{i-1}$$
$$c_i = 3(f_i - f_{i-1}) - h_i(2s'_{i-1} + s'_i)$$
$$d_i = 2(f_{i-1} - f_i) + h_i(s'_{i-1} + s'_i)$$

Thereafter we use the coefficients to calculate the cubic spline $s(x)$. With the extra conditions as periodic boundary conditions or otherwise leads to a well-defined set $\{s'_j\}_{j=0}^{n}$, and thereby a unique interpolating cubic spline.

Example 8

It exist a simple example of cubic spline. Suppose we want to make a periodic curve that is running through the four control points $(0,0), (3,1), (5,1), (8,0)$. We observe that the endpoints are joined, so we can use the periodic cubic spline. In this case the spline generates the matrix equation

$$\begin{bmatrix} 1 & & & -1 \\ 2 & 10 & 3 & \\ & 3 & 10 & 2 \\ 6 & 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} s'_0 \\ s'_1 \\ s'_2 \\ s'_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}$$

where it satisfies $s_0(x) = s_3(x)$, $s'_0(x) = s'_3(x)$ and $s''_0(x) = s''_3(x)$ at the point $x = 0$.

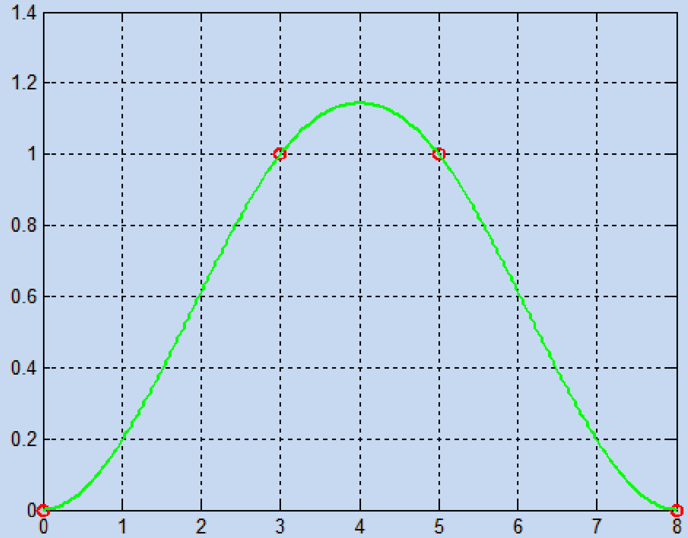


Figure 11 The periodic cubic spline with four knots in one dimensional.

It is important to remember, that we can only use the cubic spline, if the number of knots is equal or larger than three. And it is not worth to use the cubic spline, if all knots have the same value, because it gives always (not surprising) a straight line as the minimal bending energy curve.

There exists a proof that shows, that the cubic spline $s(x)$ among all functions g that are twice continuously differentiable on the interval $[a, b]$, and that interpolate a given smooth periodic function f in the points $a = x_0 < x_1 < \cdots < x_n = b$, minimizes the integral

$$E = \int_a^b g''(x)^2 dx$$

Says in another way we will show, that $\int_a^b g''(x)^2 dx \geq \int_a^b s''(x)^2 dx$. We begin to consider

$$\begin{aligned}\int_a^b (g''(x) - s''(x)\,)^2 dx &= \int_a^b g''(x)^2 dx + \int_a^b s''(x)^2 dx - 2\int_a^b g''(x)s''(x)dx \\ &= \int_a^b g''(x)^2 dx - \int_a^b s''(x)^2 dx - 2\int_a^b \big(g''(x) - s''(x)\big)s''(x)dx\end{aligned}$$

By using the partial integration in the last integral for the contribution from the ith interval reveals

$$\begin{aligned}I_i &= \int_{x_{i-1}}^{x_i} \big(g''(x) - s''(x)\big)s''(x)dx \\ &= \big[\big(g'(x) - s'(x)\big)s''(x)\big]_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} \big(g'(x) - s'(x)\big)s^{(3)}(x)dx\end{aligned}$$

We know that $s^{(3)}(x) = 6d_i$ for $x_{i-1} < x < x_i$, which is a constant, and $g(x_i) - s(x_i) = 0$ for all i because they are both interpolated to f at the points x_i, i.e. $g(x_i) = f(x_i)$ and $s(x_i) = f(x_i)$, so

$$\begin{aligned} I_i &= \left[\left(g'(x)-s'(x)\right)s''(x)\right]_{x_{i-1}}^{x_i} - 6\int_{x_{i-1}}^{x_i}\left(g'(x)-s'(x)\right)d_i dx \\ &= \left[\left(g'(x)-s'(x)\right)s''(x) - 6d_i\left(g(x)-s(x)\right)\right]_{x_{i-1}}^{x_i} \\ &= \left(g'(x_i)-s'(x_i)\right)s''(x_i) - \left(g'(x_{i-1})-s'(x_{i-1})\right)s''(x_{i-1}) \end{aligned}$$

Consequently is

$$\begin{aligned} \int_a^b \left(g''(x)-s''(x)\right)s''(x)dx &= I_1 + I_2 + \cdots + I_n \\ &= \left(g'(x_1)-s'(x_1)\right)s''(x_1) - \left(g'(x_0)-s'(x_0)\right)s''(x_0) \\ &\quad + \left(g'(x_i)-s'(x_i)\right)s''(x_i) - \left(g'(x_{i-1})-s'(x_{i-1})\right)s''(x_{i-1}) + \cdots \\ &\quad + \left(g'(x_n)-s'(x_n)\right)s''(x_n) - \left(g'(x_{n-1})-s'(x_{n-1})\right)s''(x_{n-1}) \\ &= \left(g'(x_n)-s'(x_n)\right)s''(x_n) - \left(g'(x_0)-s'(x_0)\right)s''(x_0) \end{aligned}$$

For a smooth periodic function we know that $f'(x_0) = f'(x_n)$ and $f''(x_0) = f''(x_n)$, then we can assume that $g'(x_0) = g'(x_n),\ s'(x_0) = s'(x_n)$ and $s''(x_0) = s''(x_n)$. Thus we have

$$\int_a^b \left(g''(x)-s''(x)\right)s''(x)dx = 0$$

Now we can insert this result in the reformulating of the expression of E:

$$\begin{aligned} \int_a^b g''(x)^2 dx &= \int_a^b s''(x)^2 dx + \int_a^b \left(g''(x)-s''(x)\right)^2 dx + 2\int_a^b \left(g''(x)-s''(x)\right)s''(x)dx \\ &= \int_a^b s''(x)^2 dx + \int_a^b \left(g''(x)-s''(x)\right)^2 dx \\ &\geq \int_a^b s''(x)^2 dx \end{aligned}$$

The minimum is obtained when $g''(x) - s''(x) = 0$ for $a \leq x \leq b$, i.e. $g(x) = s(x) + k_1 x + k_2$, where k_1 and k_2 are arbitrary constants. But the interpolation condition involve that $g(x_i) = s(x_i)$ for all i if and only if $k_1 = k_2 = 0$, therefore $g(x) = s(x)$.

A reason why we are saying a smooth periodic function is that we want to make a function (naturally in three dimensional) that looks like the base curve at the Möbius strips. The base curve is a smooth closed curve, it is why.

However, if the function f is not required to be smooth periodic and g is required not only to interpolate the given points, but also to satisfy the boundary conditions $g'(a) = f'(a)$ and $g'(b) = f'(b)$, then the same integral is minimized, when $g(x) = s(x)$, the interpolation cubic spline with correct boundary conitions. However also, if the function f is not required to be smooth periodic, and again the same integral is minimized, when boundary conditions of the interpolation cubic spline is natural.

We now know how to make a function from a set of knots by using the periodic cubic spline curve. Now we want to extend these ideas to arbitrary curves in two or three dimensional space.

In three dimensional the polynomial curves have the general form:

$$\boldsymbol{p}\big(x(t), y(t), z(t)\big) = \begin{pmatrix} a_1 + b_1 x(t) + c_1 x(t)^2 + d_1 x(t)^3 + \cdots \\ a_2 + b_2 y(t) + c_2 y(t)^2 + d_2 y(t)^3 + \cdots \\ a_3 + b_3 z(t) + c_3 z(t)^2 + d_3 z(t)^3 + \cdots \end{pmatrix}$$

We write three systems of linear equations for x, y and z coordinates separately. Each system is solved by following the same process as we have used in the previous. The only major difference being that we solve three linear systems instead of one with the independently variable is t. I.e. we write the system like here:

$$\begin{bmatrix} 1 & & & & -1 & \\ h_2 & 2(h_1+h_2) & h_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & h_n & 2(h_{n-1}+h_n) & & h_{n-1} \\ & 2h_n & h_n & & h_1 & 2h_1 \end{bmatrix} \begin{bmatrix} sx'_0 & sy'_0 & sz'_0 \\ sx'_1 & sy'_1 & sz'_1 \\ \vdots & \vdots & \vdots \\ sx'_{n-1} & sy'_{n-1} & sz'_{n-1} \\ sx'_n & sy'_n & sz'_n \end{bmatrix} = \begin{bmatrix} rx_0 & ry_0 & rz_0 \\ rx_1 & ry_1 & rz_1 \\ \vdots & \vdots & \vdots \\ rx_{n-1} & ry_{n-1} & rz_{n-1} \\ rx_n & ry_n & rz_n \end{bmatrix}$$

It is a point to consider. In a space, it is quite possible for a curve to join back on itself to make a closed curve. If we have the situation, we enforce C^0, C^1 and C^2 continuity where the curves joins, then we have all the equations needs without to enter any slopes.

- C^0 continuity means the two segments match values at the join.
- C^1 continuity means they match slopes at the join.
- C^2 continuity means they match curvatures at the join.

If a spline satisfies the all three conditions, the segments in the spline match torsion at the join. The definition of the torsion is determined by the slope and the curvature.

Like in the one dimensional space the elasticity of the cubic spline in the three dimensional space with the constraint of the knots will cause the strip to take the shape that minimized the energy required for bending it between the fixed points.

Making the base curve with the cubic B-spline

Our primary problem by using the interpolated technique described above is, that it can be hard to generate a base curve which gives a nonorientable (will be explained later) ruled surface like Möbius strip. To our lucky some scientists have investigated the technique further and get a fine result. The new technique is called B-splines. B-spline is a spline function that has minimal support what it regards degree, smoothness and domain partition.

The cubic spline s can expressed as a linear combination of basis splines B_{i3}, so-called cubic B-splines.

$$s(x) = \sum_i c_i B_{i,3}(x)$$

Where c_i are the control points. The definition of the basis splines is written here:

Definition 8

Let x_i be knots. We define the B-splines by the expression

$$B_{i,0}(x) = \begin{cases} 1, & x_i \leq x < x_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

And for $r = 1,2,3$:

$$B_{i,r}(x) = \frac{x - x_i}{x_{i+r} - x_i} B_{i,r-1}(x) + \frac{x_{i+r+1} - x}{x_{i+r+1} - x_{i+1}} B_{i+1,r-1}(x)$$

Here is an example of the B-splines.

Example 9

For a B-spline with the degree $r = 1$ in the interval $[x_0, x_n]$ we get

$$\begin{aligned} B_{i,1}(x) &= \frac{x - x_i}{x_{i+1} - x_i} B_{i,0}(x) + \frac{x_{i+2} - x}{x_{i+2} - x_{i+1}} B_{i+1,0}(x) \\ &= \left\{ \begin{matrix} \frac{x - x_i}{x_{i+1} - x_i}, & x_i \leq x < x_{i+1} \\ 0, & \text{otherwise} \end{matrix} \right\} + \left\{ \begin{matrix} \frac{x_{i+2} - x}{x_{i+2} - x_{i+1}}, & x_{i+1} \leq x < x_{i+2} \\ 0, & \text{otherwise} \end{matrix} \right\} \\ &= \begin{cases} 0 & x < x_i \\ \frac{x - x_i}{x_{i+1} - x_i} & x_i \leq x < x_{i+1} \\ \frac{x_{i+2} - x}{x_{i+2} - x_{i+1}} & x_{i+1} \leq x < x_{i+2} \\ 0 & x_{i+2} \leq x \end{cases} \end{aligned}$$

With $r = 2$ or more the expression becomes more complicated, then we don’t write it here, but we can illustrated the functions.

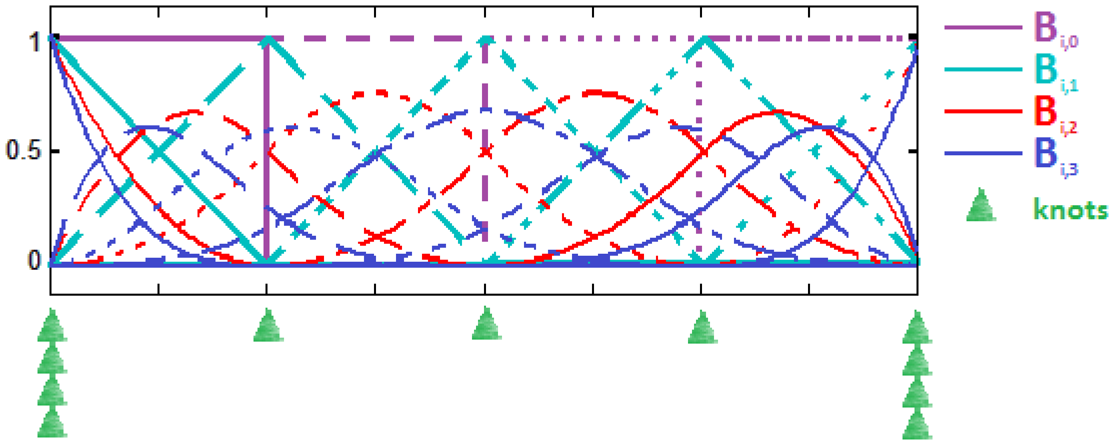


Figure 12 The B-spline with the degrees $r = 0,1,2,3$.

The degree r indicates how many knots the B-spline want to "jump over". On the figure we recognize that B-spline with $r = 1$ as a linear spline. It jumps over only a knot. For the quadratic spline it jumps over two knots, etc.

How it jumps over the knots depends on the degree and how the knots are placed relative to each other. For the cubic B-spline we can consider the shape like here:

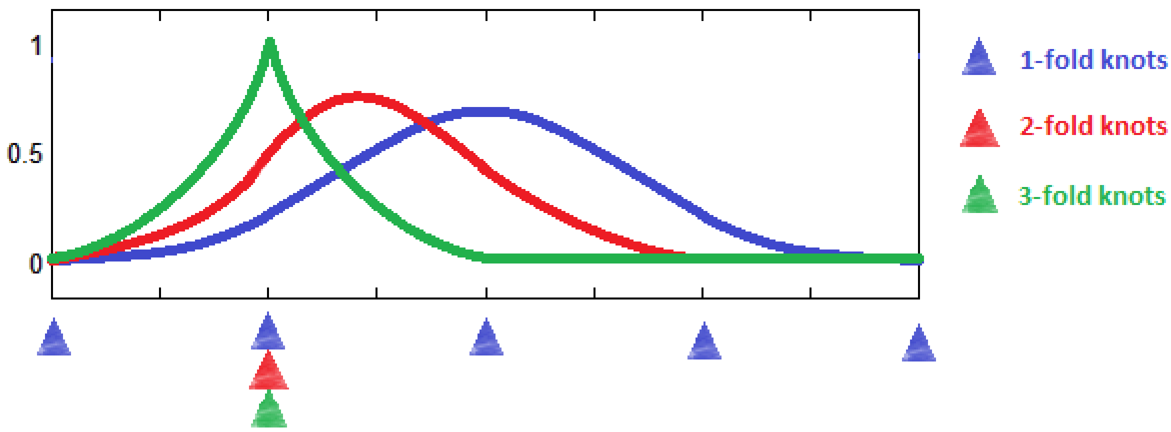


Figure 13 The cubic B-spline with a various multiplicities.

If there is 4-fold knot or more, the cubic B-spline is going to end there.

An advantage by using the cubic B-spline is, that it is easy to differentiate. The k-derivative of the B-spline is

$$B_{i,r}^{(k)}(x) = \frac{k \cdot B_{i,r-1}^{(k-1)}(x)}{x_{i+r} - x_i} + \frac{(x - x_i) \cdot B_{i,r-1}^{(k)}(x)}{x_{i+r} - x_i} - \frac{k \cdot B_{i+1,r-1}^{(k-1)}(x)}{x_{i+r+1} - x_{i+1}} + \frac{(x_{i+r+1} - x) \cdot B_{i+1,r-1}^{(k)}(x)}{x_{i+r+1} - x_{i+1}}$$

Where $k \in \mathbb{N}/\{0\}$ and $B_{i,0}^{(k)}(x) = 0$ for all i.

Example 10

For $k = 1$ we get

$$B'_{i,r}(x) = \frac{B_{i,r-1}(x)}{x_{i+r} - x_i} + \frac{(x - x_i) \cdot B'_{i,r-1}(x)}{x_{i+r} - x_i} - \frac{B_{i+1,r-1}(x)}{x_{i+r+1} - x_{i+1}} + \frac{(x_{i+r+1} - x) \cdot B'_{i+1,r-1}(x)}{x_{i+r+1} - x_{i+1}}$$

With $r = 1$ the result of the equation is

$$\begin{aligned} B'_{i,1}(x) &= \frac{B_{i,0}(x)}{x_{i+1} - x_i} + \frac{(x - x_i) \cdot B'_{i,0}(x)}{x_{i+1} - x_i} - \frac{B_{i+1,0}(x)}{x_{i+2} - x_{i+1}} + \frac{(x_{i+r+1} - x) \cdot B'_{i+1,0}(x)}{x_{i+2} - x_{i+1}} \\ &= \frac{B_{i,0}(x)}{x_{i+1} - x_i} - \frac{B_{i+1,0}(x)}{x_{i+2} - x_{i+1}} \\ &= \begin{Bmatrix} \frac{1}{x_{i+1} - x_i}, & x_i \le x < x_{i+1} \\ 0, & \text{otherwise} \end{Bmatrix} - \begin{Bmatrix} \frac{1}{x_{i+2} - x_{i+1}}, & x_{i+1} \le x < x_{i+2} \\ 0, & \text{otherwise} \end{Bmatrix} \\ &= \begin{cases} 0 & x < x_i \\ \frac{1}{x_{i+1} - x_i} & x_i \le x < x_{i+1} \\ \frac{-1}{x_{i+2} - x_{i+1}} & x_{i+1} \le x < x_{i+2} \\ 0 & x_{i+2} \le x \end{cases} \end{aligned}$$

Notice that $B_{i,r}^{(k)}(x) = 0$ for $r = 0,1,2, \dots, k - 1$. It holds for all i. Look at the illustration below to understand why.

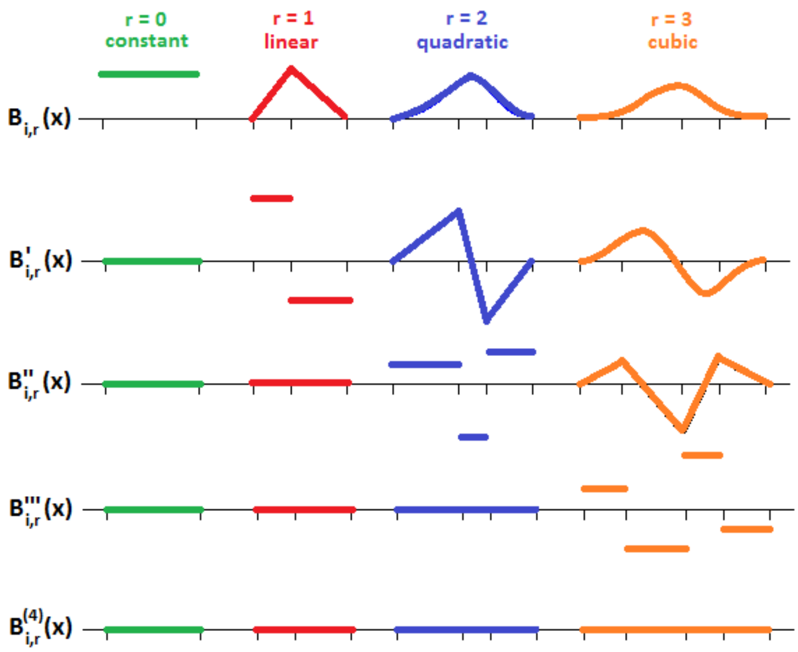


Figure 14 The B-splines of various degrees with simple knots and their derivatives.

It is illustrated an example of a cubic B-spline with various orders of the fold. The illustration tells us, that the order of the fold has no influence on the claim.

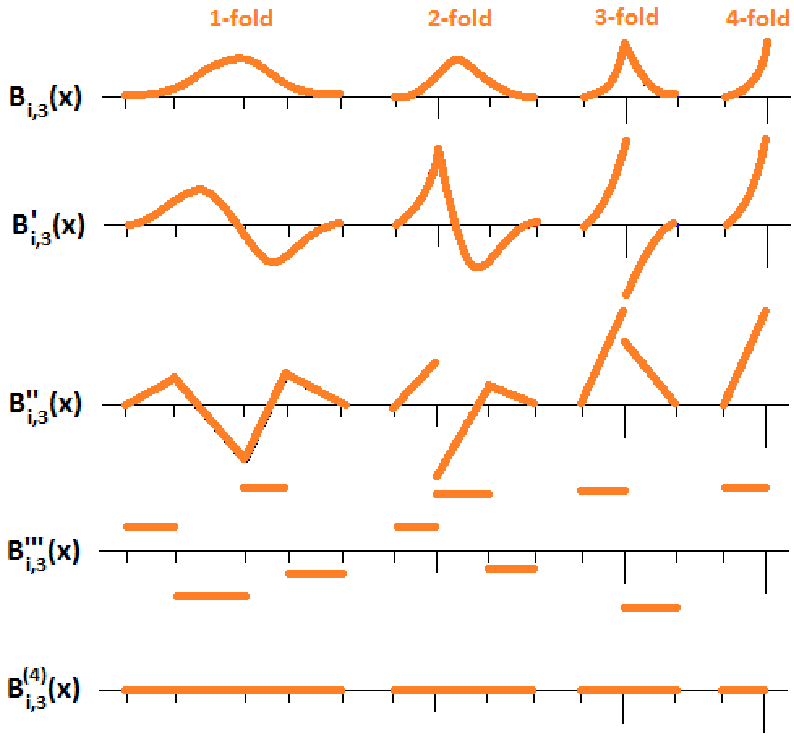


Figure 15 The cubic B-spline with a knot of various multiplicities and their derivatives.

Now we know that how the knots have the influence on the derivatives, then we take a step further in the discussion about using B-spline to generate a base curve at the Möbius strip. Which fold is the best to use for generating the curve? Is it sufficient to use simple knots during the entire path or have we to use 2-fold knots or other similarly techniques?

We wished to generate a C^0, C^1 and C^2 continuity base curve, then the simple knots will be the perfect solution, because it is the only one of the four different fold-technique, which is C^2 continuity.

It is not sufficient to use the simple knots during the entire path. The base curve at the Möbius strip is joined back on itself to make a closed curve. Therefore it is a good idea to form the B-splines then it becomes periodic and can consider as a closed ring like the figure 16. It means that the axis x is walked from x_0 to x_n and can walks further by walking again from x_0. In this way we can be sure that the B-spline is C^0, C^1 and C^2 continuity at the join.

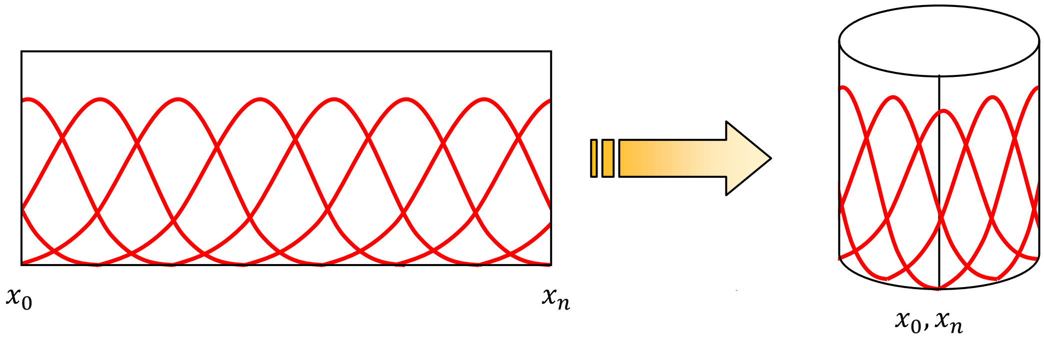


Figure 16 The periodic cubic B-spline can reshape to a closed ring.

To get the B-spline like the box to left in the figure above, the knots have to be placed uniform along the axis. The B-spline like the ring to right in the figure above satisfies the following conditions:

$$B_{-3,3}^{(k)}(x_0) = B_{n-3,3}^{(k)}(x_n)$$
$$B_{-2,3}^{(k)}(x_0) = B_{n-2,3}^{(k)}(x_n)$$
$$B_{-1,3}^{(k)}(x_0) = B_{n-1,3}^{(k)}(x_n)$$
$$B_{0,3}^{(k)}(x_0) = B_{n,3}^{(k)}(x_n) = 0$$

For $k = 0, \dots, 2$. Note that the curve $\boldsymbol{\alpha}$ is defined by

$$\boldsymbol{\alpha}(x) = \sum_{i=0}^{n-1} \boldsymbol{c}_i B_{i,3}(x)$$

Where it is given a point x belonging the intervals $x_0 \leq x_1 \leq \cdots \leq x_n$ of the curve. We may remember that $B_{i,3}(x)$ is nonzero only in the open interval $]x_i, x_{i+4}[$. I.e. for a given x in $[x_i, x_{i+1}]$ there are at most four nonzero basis splines, $B_{i-3,3}(x), B_{i-2,3}(x), B_{i-1,3}(x)$ and $B_{i,3}(x)$. At a knot there are only three nonzero B-splines. Thus we can expressed the curve as

$$\boldsymbol{\alpha}(x) = \sum_{i=-3}^{n-1} \boldsymbol{c}_i B_{i,3}(x)$$

But we compute only for the sake of the interval $x \in [x_i, x_{i+1}[$, which means we only need to include the terms with $B_{i,r-1} \neq 0$. If we investigate the definition of B-spline:

$$B_{i,r}(x) = \frac{x - x_i}{x_{i+r} - x_i} B_{i,r-1}(x) + \frac{x_{i+r+1} - x}{x_{i+r+1} - x_{i+1}} B_{i+1,r-1}(x)$$

We are going to see, that in the interval $x \in [x_i, x_{i+1}[$ the only nonzero B-spline of degree zero is $B_{i,0}(x)$. By increasing the degree we spot, the only nonzero B-spline is $B_{i-1,1}(x)$ and $B_{i,1}(x)$. For the quadratic B-spline there is only $B_{i-2,2}(x), B_{i-1,2}(x)$ and $B_{i,2}(x)$, etc. The computation of that can demonstrated excellent as figure 17.

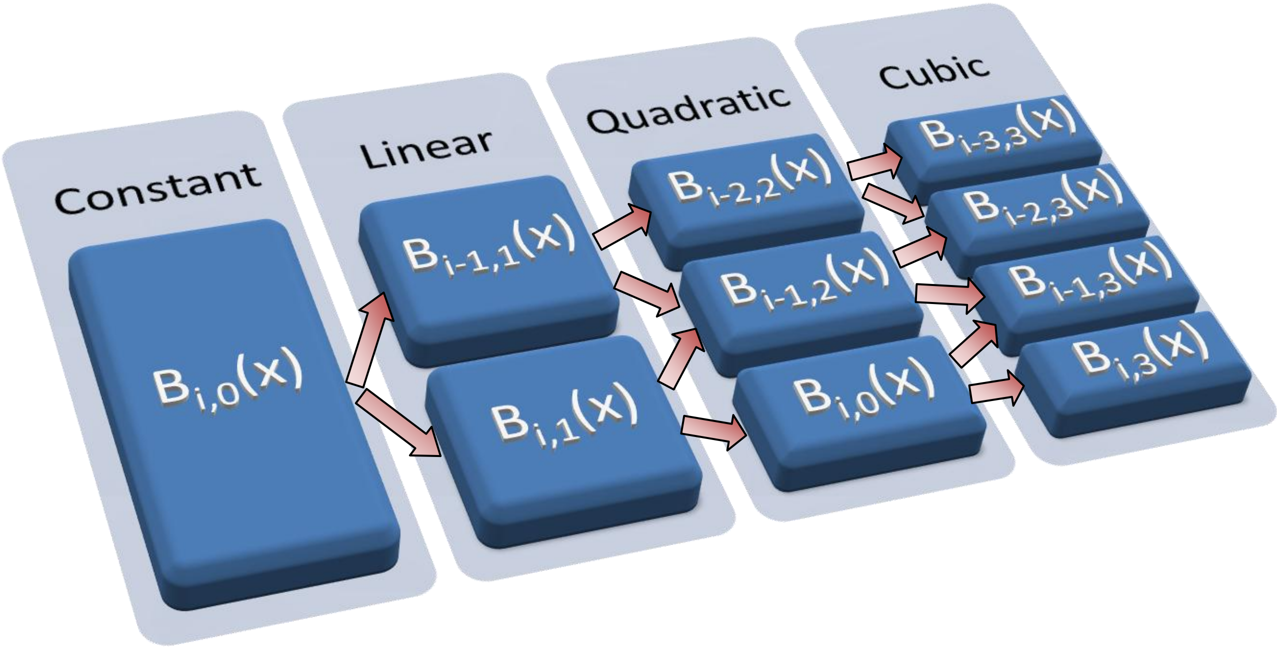


Figure 17 Nonzero values in the cubic B-splines when $x \in [x_i, x_{i+1}]$.

It is written a consequence in the example below.

Example 11

For a given $x \in [x_i, x_{i+1}[$ we find

$$\begin{aligned}
B_{i-3,3}(x) &= \frac{x_{i+1} - x}{x_{i+1} - x_{i-2}} B_{i-2,2}(x) \\
B_{i-2,3}(x) &= \frac{x - x_{i-2}}{x_{i+1} - x_{i-2}} B_{i-2,2}(x) + \frac{x_{i+2} - x}{x_{i+2} - x_{i-1}} B_{i-1,2}(x) \\
B_{i-1,3}(x) &= \frac{x - x_{i-1}}{x_{i+2} - x_{i-1}} B_{i-1,2}(x) + \frac{x_{i+3} - x}{x_{i+3} - x_i} B_{i,2}(x) \\
B_{i,3}(x) &= \frac{x - x_i}{x_{i+3} - x_i} B_{i,2}(x)
\end{aligned}$$

The example shows that the computation for $x \in [x_i, x_{i+1}[$ involves only knots $x_{i-2}, \dots, x_{i+3}$, while for $x \in [x_{n-1}, x_n[$ involves knots $x_{n-3}, \dots, x_{n+2}$. It means that for the cubic B-spline with $0 \le i \le n-1$ we need two extra knots at both ends. Remember the extra knots must satisfy $x_{-2} \le x_{-1} \le x_0$ and $x_n \ge x_{n+1} \ge x_n$. Often the extra knots are chosen such that they are placed correspondingly with x_0 and x_n. But in our case we choose

$$\begin{aligned}
x_{-2} &= x_0 - x_{n-2} \\
x_{-1} &= x_0 - x_{n-1} \\
x_{n+1} &= x_n + x_1 \\
x_{n+2} &= x_n + x_2
\end{aligned}$$

In this ways the conditions for establishing of a closed ring like figure 16 is satisfied.

There exists an general equation, which can tell us how many extra knots we need for any degree r. This equation is the property of the degree r of the B-spline

$$r \equiv m - n - 1$$

Where $m + 1$ is the number of knots and $n + 1$ is the number of control points.

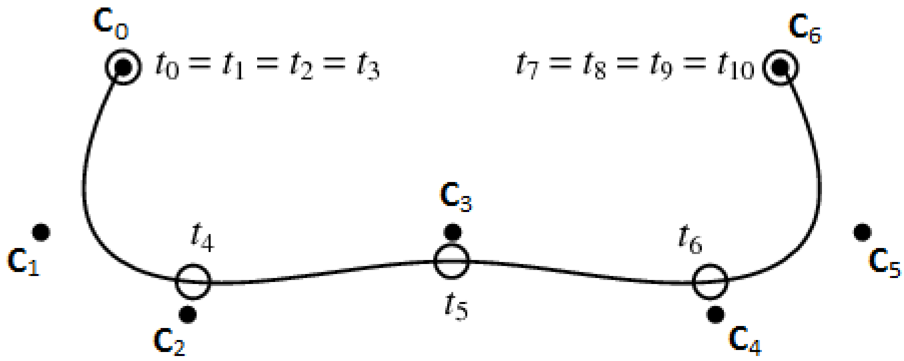


Figure 18 An example of the cubic B-spline with 7 control points and 11 knots (a non-closed curve in two dimensions).

Example 12

To get a cubic B-spline with 7 control points, we need $3 = m - 6 - 1 \Leftrightarrow m = 10$, hence 11 knots.

Note that example 12 is the same as the figure 18.

Back to our choice of the extra knots. The consequence of the choice is, that we have to choose the following control points:

$$\begin{aligned} \boldsymbol{c}_{-2} &= \boldsymbol{c}_{n-2} \\ \boldsymbol{c}_{-1} &= \boldsymbol{c}_{n-1} \\ \boldsymbol{c}_{0} &= \boldsymbol{c}_{n} \\ \boldsymbol{c}_{n+1} &= \boldsymbol{c}_{1} \\ \boldsymbol{c}_{n+2} &= \boldsymbol{c}_{2} \end{aligned}$$

to could get a closed curve. It means that now we can be sure that the curve $\boldsymbol{\alpha}$ satisfies $\boldsymbol{\alpha}(x_0) = \boldsymbol{\alpha}(x_n)$. You can see the result in the figure 22.

In the section about nonorientable we will read that we need at least 7 control points to could generate a Möbius strip. It gives us 15 knots which four of them are extra knots using to the computation.

The Möbius Strip as a Nonorientable Surface

A smooth surface is called orientable, if the positive normal direction, when given at an arbitrary point of the surface, can be continued in a unique and continuous way to the entire surface. Then a sufficiently small piece of a smooth surface is always orientable. This may not hold for entire surfaces. It is well known that the Möbius Strip is a nonorientable smooth surface, because the normal direction in every point of the strip is not unique. Now we try to find conditions for the expression, $\boldsymbol{\sigma} = \boldsymbol{\alpha} + v\left(\frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}\right)$ with the interval $u \in [0, L]$ and $v \in \left[-\frac{d}{2}, \frac{d}{2}\right]$, such that it generates a nonorientable smooth surface like the Möbius strip. For the closed and smooth base curve we require that the expression satisfies the condition respectively $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(L)$ and $\boldsymbol{\alpha}^{(n)}(0) = \boldsymbol{\alpha}^{(n)}(L)$ for every $n \in \mathbb{N}$. We can divide the surface in sufficiently small pieces, so all pieces are orientable like the figure below.

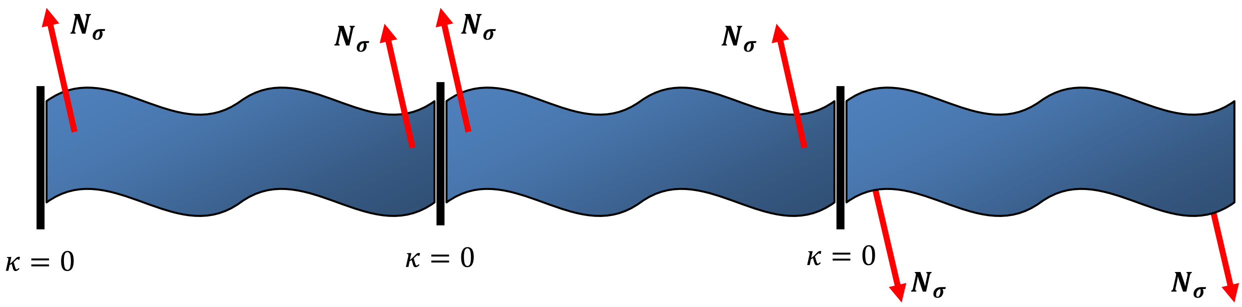


Figure 19 Möbius strip divided in many sufficiently small pieces.

For a nonorientable strip like the Möbius strips we expect that the torsion has rotated the end of the surface with $N \cdot 180$ degrees (or says in the other ways $N \cdot \pi$ in radians), where N is the odd number of the half-rotation, and let both ends join. Since the surface is smooth the consequence is the standard unit normal $\boldsymbol{N_\sigma}$ at the two endpoints on the strip has this relation:

$$\boldsymbol{N_\sigma}(0, v) = -\boldsymbol{N_\sigma}(L, -v)$$

Since the base curve is closed and smooth, we have the tangent vector at the endpoints

$$\boldsymbol{t}(0) = \frac{\dot{\boldsymbol{\alpha}}(0)}{\|\dot{\boldsymbol{\alpha}}(0)\|} = \frac{\dot{\boldsymbol{\alpha}}(L)}{\|\dot{\boldsymbol{\alpha}}(L)\|} = \boldsymbol{t}(L)$$

So the rotation generates the following conditions of the base curve

$$\boldsymbol{n}(0) = -\boldsymbol{n}(L)$$
$$\boldsymbol{b}(0) = -\boldsymbol{b}(L)$$

It is illustrated the problem in the next page and the illustration shows, how we get the three conditions above. A simplify case of the Möbius strips has the value $N = 1$.

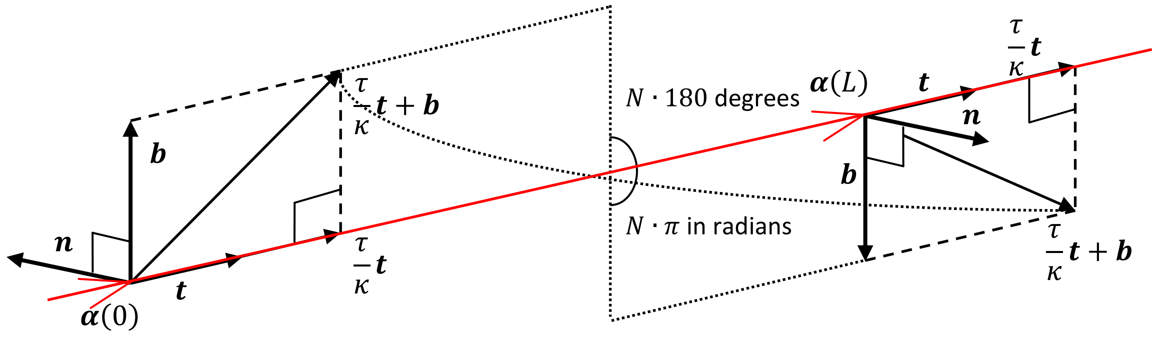


Figure 20 The rotation of the Fernet-Serret vectors at the base curve with the odd number N.

Perhaps you are tempted to think, that you can achieve this property, if the choice of the ruling $\boldsymbol{w}(u)$ changes to the ruling

$$\cos\left(\frac{N\cdot\pi}{L}\cdot u\right)\boldsymbol{w}+\sin\left(\frac{N\cdot\pi}{L}\cdot u\right)\boldsymbol{w}\times\boldsymbol{t}$$

It is added the vector $\boldsymbol{w}\times\boldsymbol{t}$, because it is perpendicular to $\boldsymbol{w}$, and so multiply the two vector with $\sin\left(\frac{N\cdot\pi}{L}\cdot u\right)$ and $\cos\left(\frac{N\cdot\pi}{L}\cdot u\right)$ respectively to get a circular motion perpendicular to the base curve $\boldsymbol{\alpha}$.

There are illustrated four examples of this property in the figure below.

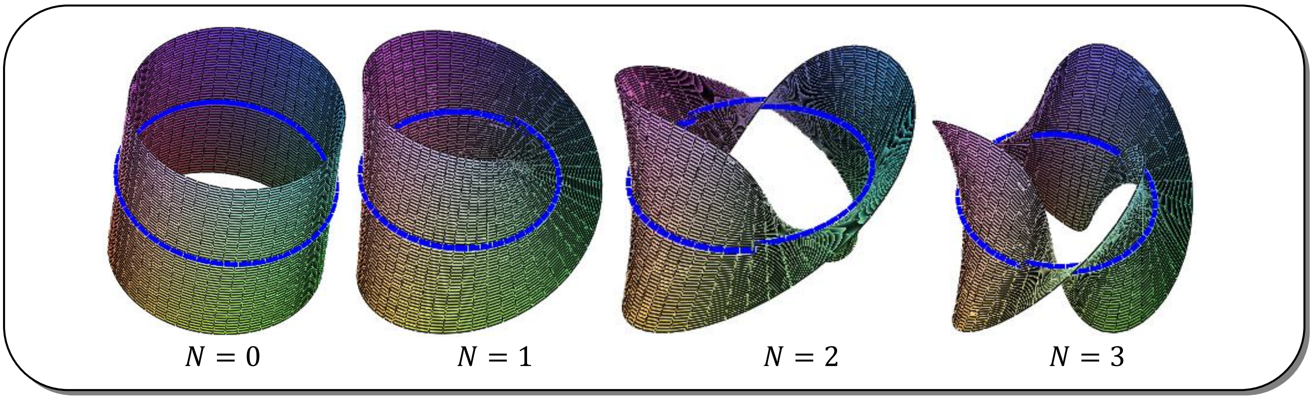


Figure 21 A surface expressed as $\boldsymbol{\sigma}(u,v)=\boldsymbol{\alpha}(u)+v\cdot\left(\cos\left(\frac{N\cdot\pi}{L}\cdot u\right)\boldsymbol{w}+\sin\left(\frac{N\cdot\pi}{L}\cdot u\right)\boldsymbol{w}\times\boldsymbol{t}\right)$.

But you have to be carefully, since we need to check about the new ruling can satisfy the two conditions:

Equation 1:

$$g\cdot(g\tau+h')=h\cdot(f\kappa+g'-h\tau)$$

Equation 2:

$$\left(1+v(f'-g\kappa)\right)g=v(f\kappa+g'-h\tau)f$$

For the ruling $\boldsymbol{w}(u) = \cos\left(\frac{N\cdot\pi}{L}\cdot u\right)\left(\frac{\tau}{\kappa}\boldsymbol{t}+\boldsymbol{b}\right) + \sin\left(\frac{N\cdot\pi}{L}\cdot u\right)\boldsymbol{n}$, (notice that $\boldsymbol{n} = \left(\frac{\tau}{\kappa}\boldsymbol{t}+\boldsymbol{b}\right)\times\boldsymbol{t}$), we achieve the following functions:

$f = \frac{\tau}{\kappa}\cos\left(\frac{N\cdot\pi}{L}\cdot u\right)$ $g = \sin\left(\frac{N\cdot\pi}{L}\cdot u\right)$ $h = \cos\left(\frac{N\cdot\pi}{L}\cdot u\right)$	$f' = \frac{\dot{\tau}\kappa - \tau\dot{\kappa}}{\kappa^2}\cos\left(\frac{N\cdot\pi}{L}\cdot u\right) - \frac{N\cdot\pi}{L}\cdot\frac{\tau}{\kappa}\sin\left(\frac{N\cdot\pi}{L}\cdot u\right)$ $g' = \frac{N\cdot\pi}{L}\cos\left(\frac{N\cdot\pi}{L}\cdot u\right)$ $h' = -\frac{N\cdot\pi}{L}\sin\left(\frac{N\cdot\pi}{L}\cdot u\right)$

The first equation $g\cdot(g\tau + h') = h\cdot(f\kappa + g' - h\tau)$ gives:

$$\tau\sin^2\left(\frac{N\cdot\pi}{L}\cdot u\right) = \frac{N\cdot\pi}{L}$$

The second equation $\big(1 + v(f' - g\kappa)\big)g = v(f\kappa + g' - h\tau)f$ gives amazing long equation, which we don't write here. But we can say, that we can by the two equations conclude that the new ruling satisfies the two conditions if and only if $N = 0$. If $N \neq 0$, it is not sure that the parameter v is the geodesic distance to the base curve. Therefore we can't guarantee we can define the region as a rectangle, when we use the new ruling for $N \neq 0$. Or says in the other ways we can't guarantee the surface with the new ruling want be developable. Here comes the cubic b-spline in the picture. In short we will take advantage of the cubic b-spline and their useful properties to making of the base curve.

From the previous section we know that a closed curve dependent of our choice for the positions of the six control points $\boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_{n-2}, \boldsymbol{c}_{n-1}, \boldsymbol{c}_n$ where $\boldsymbol{c}_0 = \boldsymbol{c}_n$. The knots at the curve are uniformly distribution.

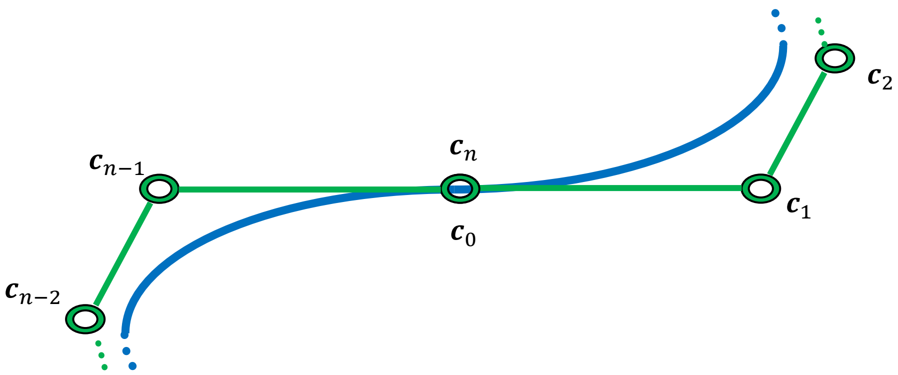


Figure 22

It will be easiest to choose the end points $\boldsymbol{c}_0$ and $\boldsymbol{c}_n$ to lie at the point $(0,0,0)$. We let the point be our default point. We can always move the coordinates translational after our desire without to change to shape of the curve.

To satisfy the condition $\boldsymbol{t}(0) = \boldsymbol{t}(L)$, we can require that the control points must satisfy

$$-(\boldsymbol{c}_{n-1} - \boldsymbol{c}_n) = \boldsymbol{c}_1 - \boldsymbol{c}_0$$

Because the tangent vector $\boldsymbol{t}$ involves only $\dot{\boldsymbol{\alpha}}$. Since $\boldsymbol{c}_0 = \boldsymbol{c}_n = \boldsymbol{0}$ we can reduce the demand to

$$-\boldsymbol{c}_{n-1} = \boldsymbol{c}_1.$$

The elegant choice will be $\boldsymbol{c}_1 = (a, 0,0) = -\boldsymbol{c}_{n-1}$, where $a \in \mathbb{R}$ is a point in the x-axis, because we can always rotate the coordinates after our desire without to change to shape of the curve.

The binormal vector at the endpoints $\boldsymbol{b}(0) = -\boldsymbol{b}(L)$ involves $\dot{\boldsymbol{\alpha}}$ and $\ddot{\boldsymbol{\alpha}}$, which means

$$-\frac{(\boldsymbol{c}_{n-2} - \boldsymbol{c}_{n-1}) + (\boldsymbol{c}_{n-1} - \boldsymbol{c}_n)}{2} = \frac{\boldsymbol{c}_2 - \boldsymbol{c}_1 + (\boldsymbol{c}_1 - \boldsymbol{c}_0)}{2}$$
$$-(\boldsymbol{c}_{n-2} - \boldsymbol{c}_n) = \boldsymbol{c}_2 - \boldsymbol{c}_0$$

Since $\boldsymbol{c}_0 = \boldsymbol{c}_n = \boldsymbol{0}$ we can reduce the demand to $-\boldsymbol{c}_{n-2} = \boldsymbol{c}_2$. Again the elegant choice will be $\boldsymbol{c}_2 = (b, c, 0) = -\boldsymbol{c}_{n-1}$, where $b, c \in \mathbb{R}$ is a point in the x- and y-axis respectively.

In short our proposal for positions of the control points is

$$\boldsymbol{c}_0 = (0,0,0) = \boldsymbol{c}_n$$
$$\boldsymbol{c}_1 = (a, 0,0) = -\boldsymbol{c}_{n-1}$$
$$\boldsymbol{c}_2 = (b, c, 0) = -\boldsymbol{c}_{n-2}$$

But it is not sufficient with 6 control points to generate a base curve for the Möbius strip. Because the curve is running through itself, if there are only 6 control points placing as our proposal. Look at the figure 23.

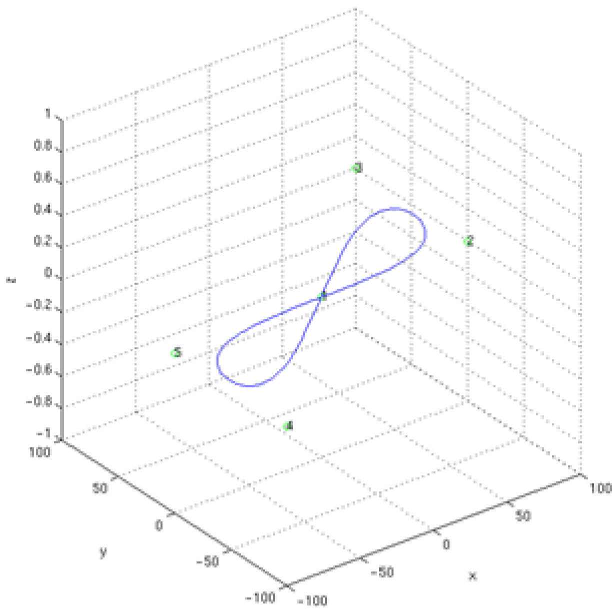


Figure 23 The cubic spline in three dimensions with only six control points.

It is not the solution we are searching after, then it is need a control point more. The new control point has to be place such that it forces the curve to don't run through itself. But where have we to place the point?

The one of the great thing about the construction of a Möbius strip with paper is, we will see a symmetry that can motivate the following expression of the base curve:

$x(u)$ is odd L-periodic function, for example $\sum_i o_i \cdot \sin\left(\frac{N\cdot\pi}{L}\cdot u\right)$

$y(u)$ is odd L-periodic function, for example $\sum_i p_i \cdot \sin\left(\frac{N\cdot\pi}{L}\cdot u\right)$

$z(u)$ is even L-periodic function, for example $\sum_i q_i \cdot \cos\left(\frac{N\cdot\pi}{L}\cdot u\right)$

The coordinates can obviously change places dependent of the choice, but here we assume just the curve has been during a so-called Procrustes Analyze. We have not to think about it now.

It is therefore often interested to build curves at the Möbius strips after the symmetrical principle. We use the knowledge to place the new control point. We can place the new control point either above or under the endpoints, $\boldsymbol{c}_4 = (0,0,d)$ where $d \in \mathbb{R}$ is a point in the x-axis, since both choices give symmetry.

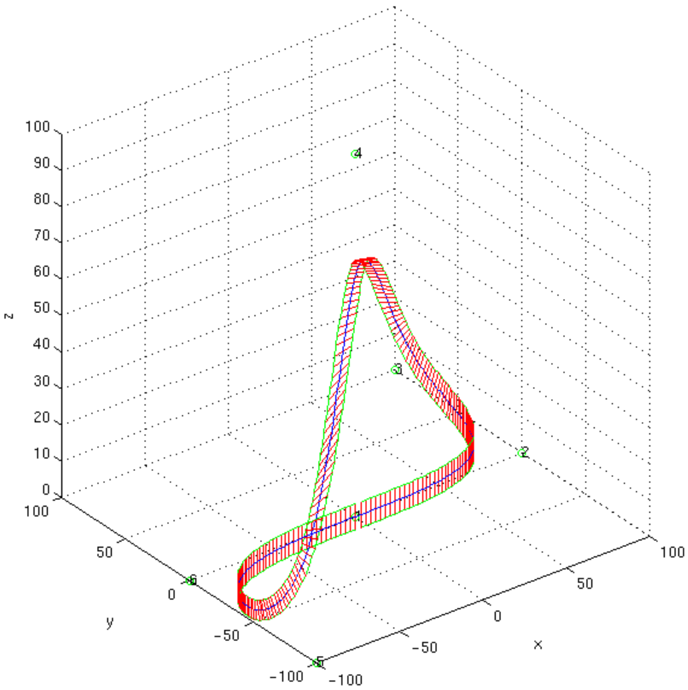


Figure 24 The Möbius strip by using the following control points,
$\boldsymbol{c}_0 = (0,0,0) = \boldsymbol{c}_n, \boldsymbol{c}_1 = (100,0,0) = -\boldsymbol{c}_{n-1}, \boldsymbol{c}_2 = (100,100,0) = -\boldsymbol{c}_{n-2}, \boldsymbol{c}_3 = (0,0,100)$

7 control points are therefore the least number we can have to make a Möbius strip. But what does it happen, if we increase the number of control points, and how we can place them so the bending energy is the least of all bending energy? It is an optimization problem and the next section will be talk about that.

The optimization of position for the control points

Now we need to think about practically problem. How can we be sure that the choice of the position of the control points such that the minimal bending energy is the least of all possibly minimal bending energy, i.e. how we can optimize the positions? Which mathematic program (MAPLE or MATLAB) is best to calculate and plot the bending energy? How we can plot the bending energy with so few inputs/parameters as possibly, so the optimization in the mathematic program will have bigger chance to find the correctly optimized solution?

First we talk about the optimization of the positions. We have the bending energy

$$\varepsilon = \frac{1}{4}\int_0^L (\tau^2 + \kappa^2)^2 \psi \cdot \ln\left(\frac{2\kappa^2\psi + d}{2\kappa^2\psi - d}\right) du$$

Where $\psi = \frac{1}{\dot{\tau}\kappa - \tau\dot{\kappa}}$. It is not indeed linear, hence not easy to find a curve, which gives the least bending energy of all curves, by using algebra method. It forces therefore us to use the weighted sum an alternative method to the integral – numerical calculation – of the bending energy. The numerical optimization can solve by mathematic programs. We use MATLAB, since the program is the best design of the two mathematic programs I know (MAPLE and MATLAB) to evaluate numerically. We tell the program, what our starting guess of control points c_i and knots t_i are. Thereafter the program can search after the optimized points. If the starting guess is already the optimized solution, then the program doesn't move them, otherwise it moves the starting control points to the optimized points. In the way we can be sure, we have the optimized solution. But we have to be careful, because we can only find the solution, if we use a good starting guess and the correctly constraints. If the starting guess is bad, we risk getting a not-complete optimized result.

The optimization becomes harder to solve and more sensitive, if we use more control points. Therefore it is suggested to use few control points as possible. From section about nonorientable we know that the least number of control points to generate a Möbius strip is 7. It is also a proposal for positions of the control points from this section we can use as our starting guess.

Now we talk about the constraints. If we compared two curves, which is identical with each other expect for scaling size - the one is simply a scaling of the second –, we want observe that the scaling have a meaning for the optimization. If the one is smaller than the second, the curvature and torsion will be larger than the second. But if the one is larger than the second, the curvature and torsion are both smaller than the second. Take now a very simply case as a circle with a radius r. Its curvature is $\kappa = 1/r$. For r walking against infinity, the curvature is going to zero.

Therefore the program will maybe be tempted to place positions infinity long away from each other in searching after the optimization. It gives infinity long the base curve at the Möbius strip, which is not the idyllic solution. We need to set a constraint in the optimization, and it is the length of the base curve must be a fixed constant. We choose, which length the strip we want to have, before we can begin optimizing. We can use the definition of the length

$$L = \int_0^1 \|\dot{\boldsymbol{\alpha}}(t)\| dt$$

as a fixed constraint in the optimization, where the base curve $\boldsymbol{\alpha}: [0,1] \to \mathbb{R}^3$.

We have also to remember to get a non complex number of the bending energy. Therefore we add a constraint in the optimization. The width of the strip is bounded by $2\kappa^2\psi > d \geq 0$.

Let us saying we have the bending energy ε. We can define a function $G(t_0, t_1, \dots, t_m, \boldsymbol{c}_0, \boldsymbol{c}_1, \dots, \boldsymbol{c}_n)$ as a bending energy of the Möbius strip by using the cubic b-splines with the starting guess knots $\{t_i\}_{i=0}^{i=m}$ and control points $\{\boldsymbol{c}_i\}_{i=0}^{i=n}$. We tell the MATLAB, that we want to minimize the function G

$$\min \varepsilon = \min G\left(\{t_i\}_{i=0}^{i=m}, \{\boldsymbol{c}_i\}_{i=0}^{i=n}\right)$$

Subject to the fixed length $L = \int_0^1 \|\dot{\boldsymbol{\alpha}}(t)\| dt$ and $2\kappa^2\psi > d \geq 0$.

Now we have to think, what MATLAB need to know before it could begin to evaluate and optimize the bending energy. Beyond the knowledge of the position of the knots and control points the optimization needs to know the length L and the width d and four functions:

$$\kappa, \tau, \dot{\kappa}, \dot{\tau}$$

if we want to optimized the position of the control points.

We can already evaluate the expression of the first two parameters, curvature and torsion:

$$\kappa = \frac{\|\ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}}\|}{\|\dot{\boldsymbol{\alpha}}\|^3}$$
$$\tau = \frac{\langle \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}, \dddot{\boldsymbol{\alpha}} \rangle}{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|^2}$$

These tell us that we can calculate the bending energy, if we have the knowledge to the three functions $\dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}}, \dddot{\boldsymbol{\alpha}}$. But what with $\dot{\kappa}$ and $\dot{\tau}$? What is the expression of the two parameters, when we use the cubic b-spline? First we try to discover an expression for the differential of the curvature.

Let the base curve $\boldsymbol{\alpha}$ at the Möbius strip be determined by the cubic b-splines with control points $\boldsymbol{c}_0, \boldsymbol{c}_1, \dots, \boldsymbol{c}_n$. We get the following

$$\boldsymbol{\alpha}(t) = \sum_{i=0}^{n} \boldsymbol{c}_i B_{i3}(t), \qquad \dot{\boldsymbol{\alpha}} = \sum_{i=0}^{n} \boldsymbol{c}_i B'_{i3}, \qquad \ddot{\boldsymbol{\alpha}} = \sum_{i=0}^{n} \boldsymbol{c}_i B''_{i3}, \qquad \dddot{\boldsymbol{\alpha}} = \sum_{i=0}^{n} \boldsymbol{c}_i B'''_{i3},$$

and

$$\boldsymbol{\alpha}^{(4)} = \boldsymbol{\alpha}^{(5)} = \cdots = 0$$

Shortly we will see why it is nice to use the cubic b-splines.

$$\kappa = \frac{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|}{\|\dot{\boldsymbol{\alpha}}\|^3} = \frac{\langle \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} \rangle^{1/2}}{\langle \dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \rangle^{3/2}} = \frac{f}{g} = \frac{\langle \boldsymbol{a}, \boldsymbol{a} \rangle^{1/2}}{\langle \boldsymbol{b}, \boldsymbol{b} \rangle^{3/2}}$$

$$\begin{aligned} \boldsymbol{a} &= \ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}} \\ \dot{\boldsymbol{a}} &= \dddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}} + \ddot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} = \dddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}} \end{aligned}$$

$$\begin{aligned} \boldsymbol{b} &= \dot{\boldsymbol{\alpha}} \\ \dot{\boldsymbol{b}} &= \ddot{\boldsymbol{\alpha}} \end{aligned}$$

$$\begin{aligned} f &= \langle \boldsymbol{a}, \boldsymbol{a} \rangle^{1/2} \\ &= \langle \ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}} \rangle^{1/2} \\ \dot{f} &= \frac{\langle \dot{\boldsymbol{a}}, \boldsymbol{a} \rangle + \langle \boldsymbol{a}, \dot{\boldsymbol{a}} \rangle}{2\langle \boldsymbol{a}, \boldsymbol{a} \rangle^{1/2}} \\ &= \frac{\langle \dot{\boldsymbol{a}}, \boldsymbol{a} \rangle}{\langle \boldsymbol{a}, \boldsymbol{a} \rangle^{1/2}} \\ &= \frac{\langle \dddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}} \rangle}{\langle \ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}} \rangle^{1/2}} \end{aligned}$$

$$\begin{aligned} g &= \langle \boldsymbol{b}, \boldsymbol{b} \rangle^{3/2} \\ &= \langle \dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \rangle^{3/2} \\ \dot{g} &= \frac{3}{2}\langle \boldsymbol{b}, \boldsymbol{b} \rangle^{1/2} \cdot \left(\langle \dot{\boldsymbol{b}}, \boldsymbol{b} \rangle + \langle \boldsymbol{b}, \dot{\boldsymbol{b}} \rangle \right) \\ &= 3\langle \boldsymbol{b}, \boldsymbol{b} \rangle^{1/2} \langle \dot{\boldsymbol{b}}, \boldsymbol{b} \rangle \\ &= 3\langle \dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \rangle^{1/2} \langle \ddot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \rangle \end{aligned}$$

$$\dot{\kappa} = \frac{\dot{f} g - f \dot{g}}{g^2} = \frac{\langle \dot{\boldsymbol{a}}, \boldsymbol{a} \rangle \langle \boldsymbol{b}, \boldsymbol{b} \rangle - 3\langle \boldsymbol{a}, \boldsymbol{a} \rangle \langle \dot{\boldsymbol{b}}, \boldsymbol{b} \rangle}{\langle \boldsymbol{a}, \boldsymbol{a} \rangle^{1/2} \langle \boldsymbol{b}, \boldsymbol{b} \rangle^{5/2}} = \frac{\langle \boldsymbol{\alpha} \times \dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}} \rangle \|\dot{\boldsymbol{\alpha}}\|^2 - 3\|\ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}}\|^2 \langle \ddot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \rangle}{\|\ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}}\| \|\dot{\boldsymbol{\alpha}}\|^5}$$

And so the differential of the torsion.

$$\tau = \frac{\langle \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}, \dddot{\boldsymbol{\alpha}} \rangle}{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|^2} = \frac{\langle \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}, \dddot{\boldsymbol{\alpha}} \rangle}{\langle \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} \rangle} = \frac{f}{g} = \frac{\langle \boldsymbol{a}, \boldsymbol{b} \rangle}{\langle \boldsymbol{a}, \boldsymbol{a} \rangle}$$

$$\begin{aligned} \boldsymbol{a} &= \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} \\ \dot{\boldsymbol{a}} &= \ddot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} + \dot{\boldsymbol{\alpha}} \times \dddot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}} \times \dddot{\boldsymbol{\alpha}} \end{aligned}$$

$$\begin{aligned} \boldsymbol{b} &= \dddot{\boldsymbol{\alpha}} \\ \dot{\boldsymbol{b}} &= \boldsymbol{\alpha}^{(4)} = \boldsymbol{0} \end{aligned}$$

N.B! We write $\boldsymbol{\alpha}^{(4)} = \boldsymbol{0}$ because of the cubic spline.

$$\begin{aligned} f &= \langle \boldsymbol{a}, \boldsymbol{b} \rangle \\ &= \langle \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}, \dddot{\boldsymbol{\alpha}} \rangle \\ \dot{f} &= \langle \dot{\boldsymbol{a}}, \boldsymbol{b} \rangle + \langle \boldsymbol{a}, \dot{\boldsymbol{b}} \rangle \\ &= \langle \dot{\boldsymbol{\alpha}} \times \dddot{\boldsymbol{\alpha}}, \dddot{\boldsymbol{\alpha}} \rangle + \langle \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}, \boldsymbol{0} \rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} g &= \langle \boldsymbol{a}, \boldsymbol{a} \rangle \\ &= \langle \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} \rangle \\ \dot{g} &= \langle \dot{\boldsymbol{a}}, \boldsymbol{a} \rangle + \langle \boldsymbol{a}, \dot{\boldsymbol{a}} \rangle \\ &= 2\langle \dot{\boldsymbol{a}}, \boldsymbol{a} \rangle \\ &= 2\langle \dot{\boldsymbol{\alpha}} \times \dddot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} \rangle \end{aligned}$$

$$\dot{\tau} = \frac{\dot{f} g - f \dot{g}}{g^2} = -\frac{2\langle \boldsymbol{a}, \boldsymbol{b} \rangle \langle \dot{\boldsymbol{a}}, \boldsymbol{a} \rangle}{\langle \boldsymbol{a}, \boldsymbol{a} \rangle^2} = -\frac{2\langle \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}, \dddot{\boldsymbol{\alpha}} \rangle \langle \dot{\boldsymbol{\alpha}} \times \dddot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} \rangle}{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|^4}$$

Therefore it is sufficient that we have the knowledge to the three functions

$$\dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}}, \dddot{\boldsymbol{\alpha}}$$

if we want to find the bending energy by using the cubic b-splines for the base curve $\boldsymbol{\alpha}$ and the Möbius strip. We don't write the bending energy as a function of the three functions here, because it will be a very long equation and not a very pretty equation.

Now we can optimize the position of the control points and get a Möbius strip which gives the least bending energy.

Results and discuss

Now we have to discuss the results, which I get by running the MATLAB.

As an example we choose the following control points

$$\boldsymbol{c}_0 = (0,0,0) = \boldsymbol{c}_6$$
$$\boldsymbol{c}_1 = (100,0,0) = -\boldsymbol{c}_5$$
$$\boldsymbol{c}_2 = (100,100,0) = -\boldsymbol{c}_4$$
$$\boldsymbol{c}_3 = (0,0,100)$$

With the control points we get a Möbius strip, which is illustrated in the figure 24. Its bending energy (before the integration) can see here:

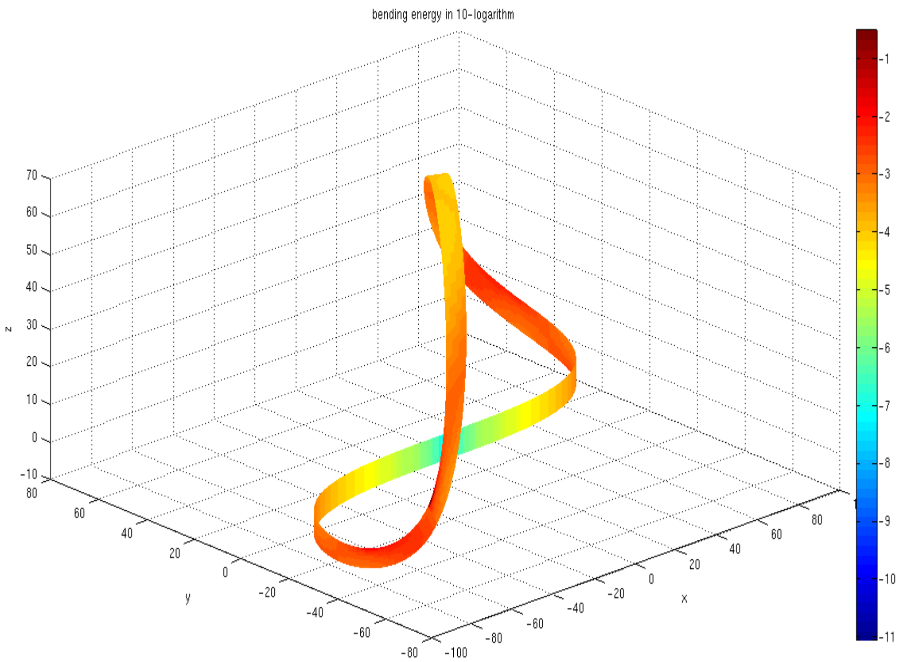


Figure 25 The bending energy in 10-logarithm for the Möbius strip.

After the integration we get $E = 3.2926 \cdot 10^{-5}$, which is very little after my opinion. The result gets me to think, how the bending energy is as a function of the width.

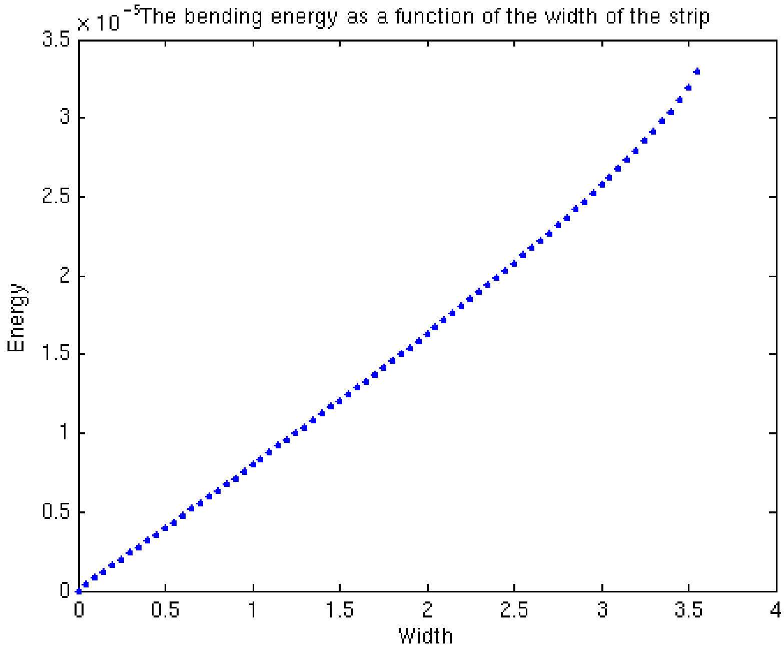


Figure 26 The bending energy as a function of the width of the strip.

As expectation the figure above shows the bending energy is proportional to the width, but the surprising thing is, that the plot seems to be almost line (until the width is about 3).

I can't get a Möbius strip with wider band like the strip to right in the figure 1. Because by the figure 27 it is noted that the minimum value of $2|\kappa^2\psi|$ is 3.5. It means that for this case we can't have d larger than 3.5. It is absolute little, since the length of this strip is about 546.338. The missing blue dots in the intervals $t \in [0,0.18]$ and $t \in [0,0.82]$ dues to the infinite. The values of $2|\kappa^2\psi|$ in the intervals are infinite.

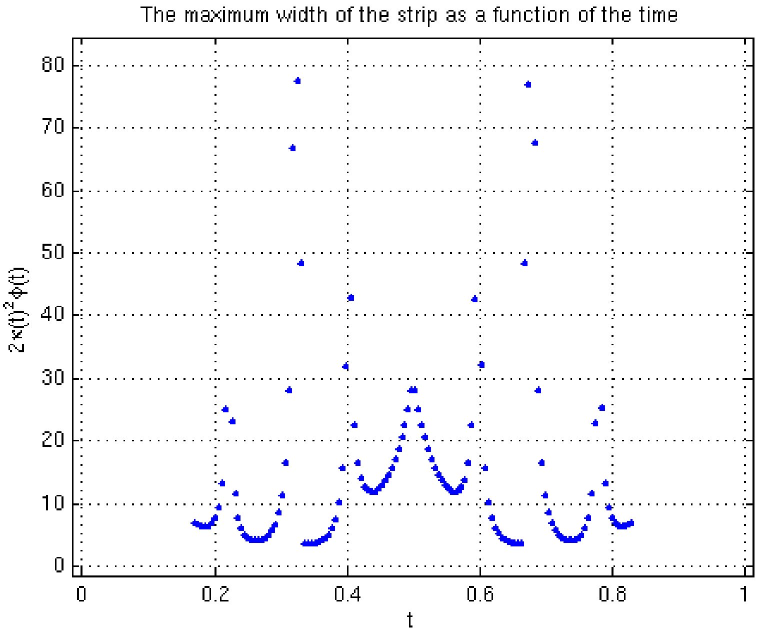


Figure 27 The value of $2|\kappa^2\psi|$ as a function of $t \in [0,1]$.

If we use the control points as our starting guess in the optimization with the length $L = 546.338$ and the width $d = 3.5$, we get:

```
Local minimum possible. Constraints satisfied.

fmincon stopped because the size of the current step is less than
the selected value of the step size tolerance and constraints were
satisfied to within the default value of the constraint tolerance.

<stopping criteria details>


The length of the strip is : 546.338
The width of the strip is : 3.5
With the bending energy E = 4.4781e-05 the optimized control points are :
points_opt =

         0  105.5584  104.2386   -0.0000 -104.2386 -105.5584         0
         0         0  105.8173   -0.0000 -105.8173         0         0
         0         0         0   51.2159         0         0         0
```

The result tells us, that in this case the optimization doesn't give a better solution, since the bending energy is $E = 4.4781 \cdot 10^{-5}$ which is a bit larger than the origanally. What does it happen, if we change the length while the width remains the same? We choose arbitrary the length $L = 195$ and $L = 200$ for the width $d = 1$. The result is

```
Local minimum possible. Constraints satisfied.

fmincon stopped because the size of the current step is less than
the selected value of the step size tolerance and constraints were
satisfied to within the default value of the constraint tolerance.

<stopping criteria details>

The length of the strip is : 195
The width of the strip is : 1
With the bending energy E = 0.00050625 the optimized control points are :
points_opt =

         0   33.3360   45.0905    0.0001  -45.0905  -33.3360         0
         0         0   30.2485    0.0001  -30.2485         0         0
         0         0         0   33.3916         0         0         0
```

```
The length of the strip is : 200
The width of the strip is : 1
With the bending energy E = 0.00044557 the optimized control points are :
points_opt =

         0   34.3668   46.0331    0.0001  -46.0331  -34.3668         0
         0         0   31.3264    0.0001  -31.3264         0         0
         0         0         0   33.6935         0         0         0
```

Not surprising the small length gives bigger bending energy.

We can try to use 8 control points with the desired length $L = 546.338$ and the width $d = 1$ for the following starting guess

$$\begin{aligned} \boldsymbol{c}_0 &= (0,0,0) = \boldsymbol{c}_7 \\ \boldsymbol{c}_1 &= (100,0,0) = -6 \\ \boldsymbol{c}_2 &= (100,100,0) = -\boldsymbol{c}_5 \\ \boldsymbol{c}_3 &= (50,50,50) \\ \boldsymbol{c}_4 &= (-50,-50,50) \end{aligned}$$

It takes a bit time before the optimization is finished.

```
Local minimum possible. Constraints satisfied.

fmincon stopped because the size of the current step is less than
the selected value of the step size tolerance and constraints were
satisfied to within the default value of the constraint tolerance.

<stopping criteria details>


The length of the strip is : 546.338
The width of the strip is : 1
With the bending energy E = 1.5769e-05 the optimized control points are :
points_opt =

0   96.1340   96.4998   49.7896  -49.7896  -96.4998  -96.1340         0
0         0   95.5635   48.8841  -48.8841  -95.5635         0         0
0         0         0   48.9982   48.9982         0         0         0
```

Here we are “lucky” to get the lower bending energy compared to only 7 control points. We are saying “lucky”, because it is used the “fail-and-try” method to find a good starting guess (using the symmetry principle).

Now we have to talk about the errors in the results.

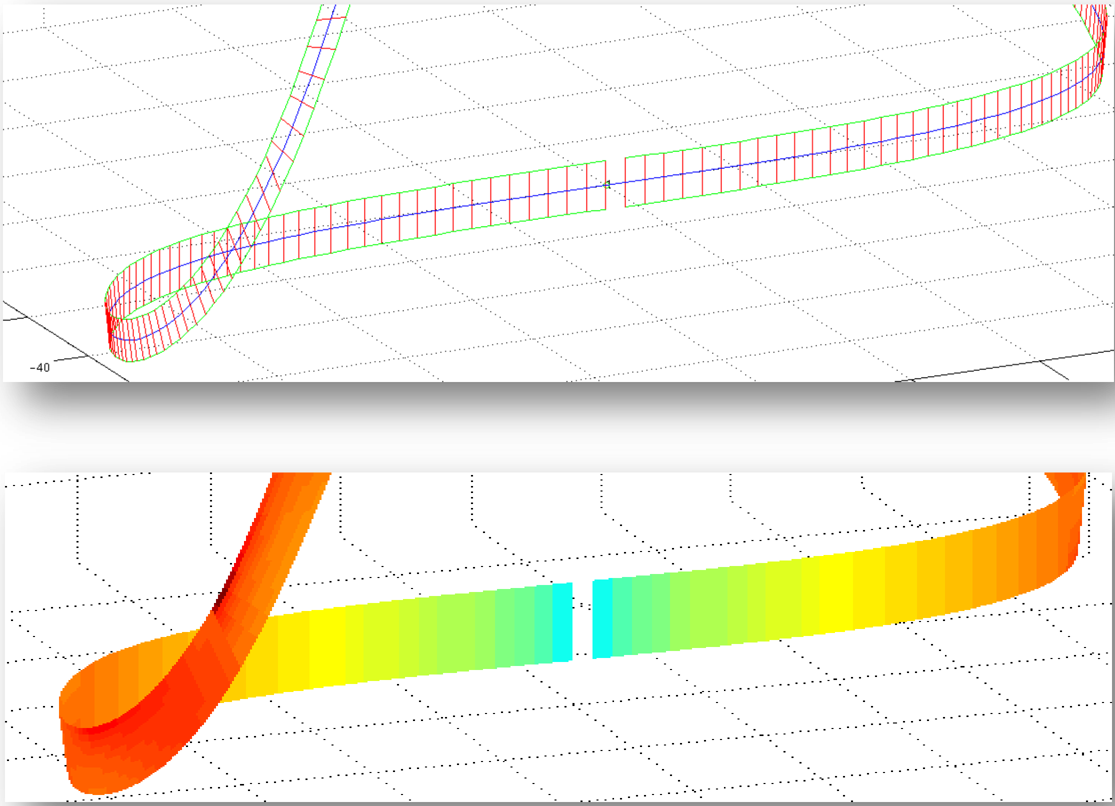


Figure 28 The missing part of the Möbius strip.

Although it is clearly sketched a closed curve (the blue curve) in the figure above, always there is missed a part of the surface. I can't explain precious why, but I am sure that it is something to do with code in MATLAB, which I have written wrong.

Conclusion

By the results we have founded in theory chapter, we conclude: A surface, that has to be a flat Möbius strip with length L and the width d, needs to satisfy the following properties:

- Belongs the ruled surface: $\boldsymbol{\sigma}(u,v) = \boldsymbol{\alpha}(u) + v \cdot \boldsymbol{w}(u)$, where $\boldsymbol{\alpha}$ is the base curve.
- Developable: $\langle \boldsymbol{\alpha}'', (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) \times \boldsymbol{\alpha}' \rangle = 0$.
- The Gaussian curvature is zero everywhere.
- Tangent vector $\boldsymbol{t}(0) = \boldsymbol{t}(L)$.
- Unit principal normal vector $\boldsymbol{n}(0) = -\boldsymbol{n}(L)$.
- Binormal vector $\boldsymbol{b}(0) = -\boldsymbol{b}(L)$.
- Express as $\boldsymbol{\sigma}(u,v) = \boldsymbol{\alpha}(u) + v\left(\frac{\tau}{\kappa}\boldsymbol{t} + \boldsymbol{b}\right)$, where κ is the curvature and τ is the torsion. Both are making of the base curve $\boldsymbol{\alpha}$.

The bending energy of the Möbius strip is: $E = \frac{1}{4}\int_0^L (\tau^2 + \kappa^2)^2 \psi \cdot \ln\left(\frac{2\kappa^2\psi + d}{2\kappa^2\psi - d}\right) du$. Small length gives larger bending energy. The bending is proportional to the width d. The width is numerical bounded by $2\kappa^2\psi$.

If it is used a cubic B-spline to generate the base curve of the Möbius strip, then the control points of the cubic spline must satisfy following

- Have least 7 control points (more control points – more sensitive)
- $\boldsymbol{c}_0 = (0,0,0) = \boldsymbol{c}_n$
- $\boldsymbol{c}_1 = (a,0,0) = -\boldsymbol{c}_{n-1}$
- $\boldsymbol{c}_2 = (b,c,0) = -\boldsymbol{c}_{n-2}$
- $\boldsymbol{c}_3 = (0,0,d)$ if we use only 7 control points, otherwise other places.
- Symmetric positions like here:

$x(u)$ is odd L-periodic function, for example	$\sum_i o_i \cdot \sin\left(\frac{N\cdot\pi}{L} \cdot u\right)$
$y(u)$ is odd L-periodic function, for example	$\sum_i p_i \cdot \sin\left(\frac{N\cdot\pi}{L} \cdot u\right)$
$z(u)$ is even L-periodic function, for example	$\sum_i q_i \cdot \cos\left(\frac{N\cdot\pi}{L} \cdot u\right)$

- The extra knots and control points

$$\begin{aligned} x_{-2} &= x_0 - x_{n-2} \\ x_{-1} &= x_0 - x_{n-1} \\ x_{n+1} &= x_n + x_1 \\ x_{n+2} &= x_n + x_2 \end{aligned} \qquad \begin{aligned} \boldsymbol{c}_{-2} &= \boldsymbol{c}_{n-2} \\ \boldsymbol{c}_{-1} &= \boldsymbol{c}_{n-1} \\ \boldsymbol{c}_{n+1} &= \boldsymbol{c}_1 \\ \boldsymbol{c}_{n+2} &= \boldsymbol{c}_2 \end{aligned}$$

The optimization of the position for the control points is dependent on a fixed length $L = \int \|\dot{\boldsymbol{\alpha}}(t)\| dt$ and $2\kappa^2\psi > d \geq 0$.

By the results we have founded in running of MATLAB program we can conclude that it works, although there is a place for improvement. The biggest problem in the results is, that it is not succeeded yet to get a very wide band like the strip to right in the figure 1.

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Appendix A: MATLAB-code

```
% Test no. 2
%
% Find the surface by using the control points at the cubic b-spline
%
% Written by: Coilin P. Boylan Jeritslev, 28th June 2011...

clear all; clc;

r = 100;

x = [  0  r  r  0 -r -r  0]; % control points in x-coordinat
y = [  0  0  r  0 -r  0  0]; % control points in y-coordinat
z = [  0  0  0  r  0  0  0]; % control points in z-coordinat

points = [x;y;z];

knots = linspace(0,1,size(points,2));

d = 5; % width of the band
i = 0;

steplength = d/100;

for s = 0:steplength:d
    i= i+1;

    [E(i),f,t,n,b,kappa,tau,ddE(:,:,i),time,Length,border] =
energy_bending2(points,s,knots);

    % If we get a complex number in the expression of the energy,
    % we stop the loop.
    if isreal(E) ~= 1
        E(i) = [];
        ddE(:,:,i) = [];
        s = s-steplength;
        break
    end
end

myplot(E,f,t,n,b,kappa,tau,ddE(:,:,i-1),time,Length,border,s)
```

```
function [E,f,t,n,b,kappa,tau,ddE,time,Length,border] =
energy_bending2(points,d,knots)

n = 200; [pp,time] = my_cubic_bspline(points,knots,n,'periodic');

figure(1)
plot3(points(1,:),points(2,:),points(3,:),'go'), hold on

for i = 1:size(points,2)-1
    text(points(1,i),points(2,i),points(3,i),num2str(i))
end

f = pp(:,:,1);

plot3(f(1,:),f(2,:),f(3,:)), axis equal square,
xlabel('x'), ylabel('y'), zlabel('z')
title('The surface with the blue centerline and the red rulings')
hold off

d1f = pp(:,:,2);
d2f = pp(:,:,3);
d3f = pp(:,:,4);

l(1) = norm(d1f(:,1),2);
Length(1) = 0;

for i = 2:size(f,2)
    l(i) = norm(d1f(:,i),2);
    Length(i) = trapz(time(1:i),l(1:i));
end

for i = 1:size(f,2)
    c1(1:3,i) = cross(d2f(:,i),d1f(:,i));
    c2(1:3,i) = cross(d1f(:,i),d3f(:,i));

    t(1:3,i) = d1f(:,i)/norm(d1f(:,i),2);
    b(1:3,i) = -c1(:,i)/norm(-c1(:,i),2);
    n(1:3,i) = cross(b(:,i),t(:,i));

    kappa(i)  = norm(c1(:,i),2)/norm(d1f(:,i),2)^3;
    dkappa(i) = (dot(-c2(:,i),c1(:,i)).*norm(d1f(:,i),2)^2. -
3*norm(c1(:,i),2)^2
.*dot(d2f(:,i),d1f(:,i)))/(norm(d1f(:,i),2)^5*norm(c1(:,i),2));

    tau(i)  =   dot(-c1(:,i),d3f(:,i))./norm(-c1(:,i),2)^2;
    dtau(i) =   -2*dot(-c1(:,i),d3f(:,i))*dot(c2(:,i),-c1(:,i))/norm(-
c1(:,i),2)^4;

    phi(i) = 1 / (dtau(i) * kappa(i) - tau(i) * dkappa(i));

    if isnan(phi(i))
        phi(i) = Inf;
    end

    border(i) = 2 * abs(kappa(i) * phi(i));
```

```
    if isnan(border(i))
        border(i) = Inf;
    end

    v = linspace(-d/2,d/2,11);

    for j = 1:length(v)
        ddE(j,i) = kappa(i)^2 * ((tau(i)/kappa(i))^2 + 1)^2 /
(1+v(j)/(kappa(i)*phi(i)));
    end

    bdE(i) = trapz(v,ddE(:,i));

    dE(i) = (tau(i)^2 +
kappa(i)^2)^2*phi(i)*log((2*kappa(i)*phi(i)+d)/(2*kappa(i)*phi(i)-d));

    if isnan(dE(i))
        dE(i) = 0;
    end
end

E = 1/4 * trapz(time,dE);


end
```

```
function [s,t] = my_cubic_bspline(c,knots,n,varargin)

C = c; KNOTS = knots;

for i = 2:length(knots)
    if knots(i) < knots(i-1)
        error('the knots must not be decreasing')
    end
end

if nargin > 3
    if strcmp(varargin{1},'periodic')
        x = [ones(1,2)*(knots(end-2)-1) knots(end-2:end-1)-1 knots 1+knots(2:3)
ones(1,2)*(1+knots(3))];
        c = [c(:,end-2:end-1) c c(:,2:3)];
    else
        x = [zeros(1,3) knots ones(1,3)];
        c = [c(:,1) c(:,:) c(:,end)];
    end
else
    x = [zeros(1,3) knots ones(1,3)];
    c = [c(:,1) c(:,:) c(:,end)];
end

t = linspace(0,1+1/(n+1),n+1);

for i = 1:length(x)-1
    for j = 1:n
        if t(j) >= x(i) && t(j) < x(i+1)
            B(i,j,1,1)= 1;
        else
            B(i,j,1,1:4)= 0;
        end
    end
end

B1 = zeros(size(B));
B2 = zeros(size(B));

for k = 1:4
    for r = 2:4
        for i = 1:length(x)-r
            for j = 1:n

                B1(i,j,r,k) = (t(j)-x(i))/(x(i+r-1)-x(i))*B(i,j,r-1,k);
                B2(i,j,r,k) = (x(i+r)-t(j))/(x(i+r)-x(i+1))*B(i+1,j,r-1,k);

                if k > 1
                    B1(i,j,r,k) =  (k-1)*B(i,j,r-1,k-1)/(x(i+r-1)-x(i)) +
B1(i,j,r,k);
                    B2(i,j,r,k) = -(k-1)*B(i+1,j,r-1,k-1)/(x(i+r)-x(i+1)) +
B2(i,j,r,k);
                end


                if isnan(B1(i,j,r,k))
                    B1(i,j,r,k) = 0;
```

```
                end
                if isnan(B2(i,j,r,k));
                    B2(i,j,r,k) = 0;
                end
                    B(i,j,r,k) = B1(i,j,r,k) + B2(i,j,r,k);

            end
        end
    end
end

%for k = 1:4
%    figure
%    for r = 1:4
%        subplot(2,2,r)
%        plot(t(1:end-1),B(1:end,:,r,k),'r','LineWidth',2)
%        xlim([0 1]),% axis equal
%        grid on
%        title(['B_{i,' num2str(r-1),'}^{ (' num2str(k-1),') }'])
%    end
%    hold off
%end

for k = 1:4
    for j = 1:n
        p = zeros(size(c,1),1);
        for i = 1:length(c)
            p = p + c(:,i)*B(i,j,4,k);
        end
        s(:,j,k) = p;
    end
end

t = t(1:end-1);

%for i = 1:size(c,1)
%    figure
%    plot(t,s(i,:,1)), hold on, plot(KNOTS,C(i,:),'go')
%    grid on
%    title('Spline function by using the base functions and control points')
%end
```

```
% Test no. 3
%
% Optimize the surface with the given length and width by using
% the starting guess position to the control points

clear all; clc;

d = 1;

intial_positions = [100;100;100; 50;50;50; -50;-50;50];

knots = linspace(0,1,floor(length(intial_positions)/3)-1+6);

options = optimset('Algorithm','interior-point','TolX',1.00E-02);

[positions_opt,E] =
fmincon(@energy_bending3,intial_positions,[],[],[],[],[],[],@mycon,options,d,kno
ts);

if isreal(E) == 1
    i = 0;

    for s = 0:d/100:d
        i= i+1;
        [E(i),f,t,n,b,kappa,tau,ddE,time,Length,border,points_opt] =
energy_bending3(positions_opt,s,knots);
    end


    fprintf('\n')
    fprintf(['The length of the strip is : ', num2str(Length(end)), '\n'])
    fprintf(['The width of the strip is : ', num2str(s), '\n'])
    fprintf(['With the bending energy E = ', num2str(E(end)),' the optimized
control points are : ']), points_opt
    fprintf('\n')

    myplot(E,f,t,n,b,kappa,tau,ddE,time,Length,border,s)
end
```

```
function [E,f,t,n,b,kappa,tau,ddE,time,Length,border,points] =
energy_bending3(x,d,knots)

points = zeros(3,(length(x)-3)/3+6);

points(:,[1 end  ]) = [0 0;0 0;0 0];
points(:,[2 end-1]) = [x(1) -x(1);0 0;0 0];
points(:,[3 end-2]) = [x(2) -x(2);x(3) -x(3);0 0];

if length(x) > 3
    for i = 4:length(x)
        points(i-floor((i-1)/3)*3,floor((i-1)/3)+3) = x(i);
    end
end

n = 200; [pp,time] = my_cubic_bspline(points,knots,n,'periodic');

figure(1)
plot3(points(1,:),points(2,:),points(3,:),'go'), hold on

for i = 1:size(points,2)-1
    text(points(1,i),points(2,i),points(3,i),num2str(i))
end

f = pp(:,:,1);

plot3(f(1,:),f(2,:),f(3,:)), axis equal square,
xlabel('x'), ylabel('y'), zlabel('z')
title('The surface with the blue centerline and the red rulings')
hold off

d1f = pp(:,:,2);
d2f = pp(:,:,3);
d3f = pp(:,:,4);

l(1) = norm(d1f(:,1),2);
Length(1) = 0;

for i = 2:size(f,2)
    l(i) = norm(d1f(:,i),2);
    Length(i) = trapz(time(1:i),l(1:i));
end

for i = 1:size(f,2)
    c1(1:3,i) = cross(d2f(:,i),d1f(:,i));
    c2(1:3,i) = cross(d1f(:,i),d3f(:,i));

    t(1:3,i) = d1f(:,i)/norm(d1f(:,i),2);
    b(1:3,i) = -c1(:,i)/norm(-c1(:,i),2);
    n(1:3,i) = cross(b(:,i),t(:,i));

    kappa(i)  = (norm(c1(:,i),2)/norm(d1f(:,i),2)^3);
    dkappa(i) = (dot(-c2(:,i),c1(:,i)).*norm(d1f(:,i),2)^2. -
3*norm(c1(:,i),2)^2
.*dot(d2f(:,i),d1f(:,i)))/(norm(d1f(:,i),2)^5*norm(c1(:,i),2));
```

```
    tau(i)  =    dot(-c1(:,i),d3f(:,i))./norm(-c1(:,i),2)^2;
    dtau(i) =    -2*dot(-c1(:,i),d3f(:,i))*dot(c2(:,i),-c1(:,i))/norm(-
c1(:,i),2)^4;

    phi(i) = 1 / (dtau(i) * kappa(i) - tau(i) * dkappa(i));

    if isnan(phi(i))
        phi(i) = Inf;
    end

    border(i) = 2 * abs(kappa(i) * phi(i));

    if isnan(border(i))
        border(i) = Inf;
    end

    v = linspace(-d/2,d/2,11);

    for j = 1:length(v)
        ddE(j,i) = kappa(i)^2 * ((tau(i)/kappa(i))^2 + 1)^2 /
(1+v(j)/(kappa(i)*phi(i)));
    end

    dE(i) = (tau(i)^2 +
kappa(i)^2)^2*phi(i)*log((2*kappa(i)*phi(i)+d)/(2*kappa(i)*phi(i)-d));

    if isnan(dE(i))
        dE(i) = 0;
    end
end

E = 1/4 * trapz(time,dE);

end
```

```
function [c,ceq] = mycon(x,s,knots)
% The conditions to the optimization of the Möbius strip
%
% Input:
% x     = the initial positions to the control points
% s     = the desired width of the strip
% knots = the initial knots to the initial positions.
%
% Output:
% c     = nonlinear inequalities at x
% ceq   = nonlinear equalities at x
%
% Written by: Coilin P. Boylan Jeritslev, 28th June 2011...

% How long the desired length of the strip must be? 195? 200?
wishlength = 546.338;

[E,f,t,n,b,kappa,tau,dE,time,Length,border,points] =
energy_bending3(x,s,knots);

c = [];

ceq = [Length(end) - wishlength; ~isreal(E)];
```

```
function myplot(E,f,t,n,b,kappa,tau,ddE,time,Length,border,s)

k = size(ddE,1);
v = linspace(0,s,floor(k/2));

figure
plot(linspace(0,s,length(E)),E,'.')
xlabel('Width')
ylabel('Energy')
title('The bending energy as a function of the width of the strip')

figure
plot(time,border,'.')
xlabel('t')
ylabel('2\kappa(t)^2\phi(t)')
title('The maximum width of the strip as a function of the time')
grid on

wx = tau./kappa .* t(1,:) + b(1,:);
wy = tau./kappa .* t(2,:) + b(2,:);
wz = tau./kappa .* t(3,:) + b(3,:);

figure(1)
hold on
plot3(f(1,:)-s/2*wx,f(2,:)-s/2*wy,f(3,:)-s/2*wz,'g')
plot3(f(1,:)+s/2*wx,f(2,:)+s/2*wy,f(3,:)+s/2*wz,'g')
plot3([1;1]*f(1,:)+[-1;1]*s/2*wx,[1;1]*f(2,:)+[-1;1]*s/2*wy,[1;1]*f(3,:)+[-
1;1]*s/2*wz,'r')
grid on

v = linspace(-s,s,k);

figure
hold on
surf(ones(k,1)*f(1,:)+v'*wx,ones(k,1)*f(2,:)+v'*wy,ones(k,1)*f(3,:)+v'*wz,log10(
ddE),'LineStyle','none')
xlabel('x'), ylabel('y'), zlabel('z')
title('bending energy in 10-logarithm')
colorbar
grid on
```